Stable Classification of 4-manifolds

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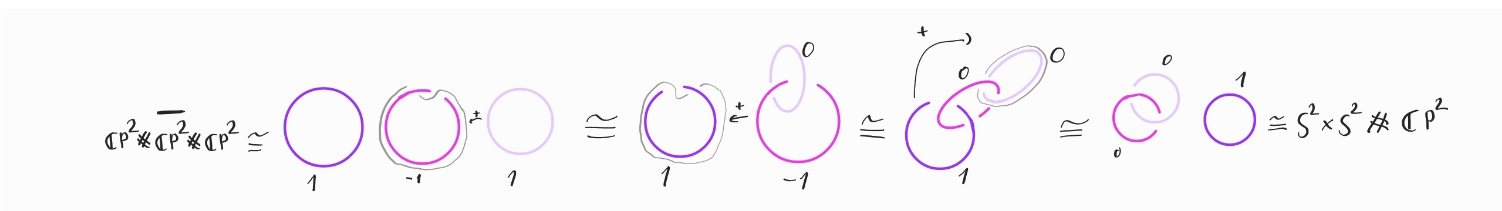
Introduction

Surgery theory has been in the forefront in the classification of smooth, PL and topological manifolds. It was sparked with Milnors's 1961 discovery of *exotic spheres* and later developed by Milnor, Browder, Kervaire, Sullivan, Wall and many others for dimensions ≥ 5. In the 1980's Freedman, Quinn and others discovered that surgery in *topological category* also works in dimension 4 for so-called *good* fundamental groups(for example f.g. abelian, finite and subexponential growth groups are good). One successful approach in the smooth category is classification of manifolds up to a stable diffeomorphism. We call two manifolds *stably diffeomorphic* if they become diffeomorphic after some number of connected sums with $S^2 \times S^2$, in other words $M \#^r S^2 \times S^2 \cong N \#^s S^2 \times S^2$.There is a variant of this question, where one allows connected sum with $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$(reversed orientation). An easy observation is

Lemma 1. *Stable diffeomorphism(homeomorphism) implies $\mathbb{C}P^2$-stable diffeomorphism(homeomorphism)*

This follows from the following isomorphism(in the Kirby calculus notation):



Stabilisation with $S^2 \times S^2$

For a 4-dimensional manifold, we can say the following about some homotopy invariants under stabilisation with $S^2 \times S^2$:

- The fundamental group $\pi_1(M \# S^2 \times S^2) \xrightarrow{\cong} \pi_1(M)$
- Consider $\pi_2(M)$ as a $\mathbb{Z}[\pi_1(M)] := \mathbb{Z}[\pi]$ module. Then, assuming π is finite, we get $\pi_2(M \#^r S^2 \times S^2) \cong \pi_2(M) \oplus \mathbb{Z}[\pi]^{2r}$. Furthermore, the $\pi_1(M)$*-equivariant intersection* form $\pi_2(\overline{M}) \times \pi_2(\overline{M}) \to \mathbb{Z}\pi$ changes by adding the hyperbolic form on $\mathbb{Z}[\pi] \oplus \mathbb{Z}[\pi]^*$
- The Postnikov 2-stage of M is given by a triple $(\pi, \pi_2(M), k)$ were $k \in H^1(\pi_1; \pi_2(M))$ is a k-invariant. Under an r-fold stabilisation the Postnikov 2-type of $M \#^r S^2 \times S^2$ is characterised by $(\pi, \pi_2 \oplus \mathbb{Z}[\pi]^{2r}, i_*(k))$ for $i : \pi_2(M) \to \pi_2(M) \oplus \mathbb{Z}[\pi]^{2r}$, the inclusion of coefficients.
- For the Euler characteristics of even dimensional manifolds we have $\chi(M \# N) + 2 = \chi(M) + \chi(N)$, so $\chi(M \#^r S^2 \times S^2) = \chi(M) + r(4-2)$. Specifically if $M \#^r S^q \times S^q \cong N \#^s S^q \times S^q$, then $r - s = \frac{\chi(M) - \chi(N)}{2}$.
- The signature of an oriented manifold is constant under stabilisation.

Normal m-types

- For a fibration $\xi : B \to BO$, the lift of the stable normal bundle, such that the following diagram commutes is called a *B-structure.*

$$\begin{array}{ccc} & & B \\ & \bar{\nu} \nearrow & \downarrow \xi \\ M & \xrightarrow{\nu} & \mathrm{BO} \end{array}$$

- A fibration $B \xrightarrow{\xi} BO$ is *m-universal* if it is a $(m+1)$-coconnected (on π_i it is an isomorphism for $i > m+1$, and an injection for $i = m+1$)
- A B-structure like above, for $\xi : B \to \mathrm{BO}$ a fibration, such that $\bar{\nu}$ is $(m+1)$-connected is called an *m-smoothing* of the manifold M
- An m-universal fibration with a normal m-smoothing of a manifold M is called a *normal m-type* of M.

Note 2. *For a normal bundle $\nu : M \to \mathrm{BO}$, the normal type $B \to \mathrm{BO}$ is a stage in its Moore-Postnikov tower.*

References:

[Kre99] Matthias Kreck. Surgery and duality. *Annals of Mathematics*, 149:707–754, 1999.

[Tei92] Peter Teichner. *Topological 4-manifolds with finite fundamental group.* PhD thesis, University of Mainz, 1992. Shaker Verlag.

[Tei93] Peter Teichner. On the signature of four-manifolds with universal cover spin. *Math. Ann. 295*, pages 745–759, 1993.

Normal 1-types

One of the main ingredients are the following bordism groups

Definition 3. *Let ξ be a fibration. Then let $\Omega_*(\xi)$ be bordism classes of smooth manifolds, with a B-structure*

It is a result of the Theorem 5 that any B- structure on M^m is B-bordant to a $\lfloor \frac{m}{2} \rfloor$-smoothing.

For a fibration $\xi : B \to BO$ define set $\mathrm{NSt}_{2q}(\xi)$ to be $(q-1)$-smoothings $(M, \bar{\nu} : M \to B)$ up to the diffeomorphism of manifolds making the following diagram commute up to homotopy:

$$\begin{array}{ccc} M' & & \\ f \uparrow & \searrow \bar{\nu}' & \\ M & \xrightarrow{\bar{\nu}} & B \end{array}$$

Under above conditions and for $q \geq 2$ we have the following theorem:

Theorem 4. *[Kre99] The obvious map $\mathrm{NSt}_{2q}(\xi) \to \Omega_{2n}(\xi)$ is a bijection.*

Remark. The theorem an easy consequence of Thm C, loc. cit., where the requirement for the same Euler characteristics is replaced by allowing different number of summants on both sides and the fact that bordism doesn't change the parity of the Euler characteristics

One problem is that, for a given manifold M, we can have multiple fibrations ξ, where the manifolds has a $(q-1)$-smoothing. But if we require ξ to be $(q-1)$-universal, there is up to homotopy a single one, justifying the name.

Modified surgery

The following theorem illustrates, why a B-structure for a manifold M is as helpful as a degree one normal map in classical surgery, in accomplishing *surgery bellow the middle dimension.*

Theorem 5. *[Kre99] Let $\xi : B \to BO$ with B connected with finite $\lfloor \frac{m}{2} \rfloor$-skeleton be a fibration and let $\bar{\nu}u : M \to B$ be a B-structure on an m-dimensional manifold M. Then there is a manifold and a B-structure $(M', \bar{\nu}')$ which is ξ-bordant to $(M, \bar{\nu})$, such that $\bar{\nu}'$ is an $\lfloor \frac{m}{2} \rfloor$ smoothing ($\lfloor \frac{m}{2} \rfloor$-connected)*

Main Idea Assume the map $\bar{\nu} : M \to B$ is $(k-1)$-connected. To make it k-connected, assuming we are still bellow the middle dimension we need to use surgery to kill elements in $\pi_k(\bar{\nu})$, which amounts to killing coker of $\pi_k(M) \to \pi_k(B)$ and in $\ker : \pi_{k+1}(M) \to \pi_{k+1}(B)$.

- Coker: A representative of an element in $f \in \mathrm{coker}(\pi_k(M) \to \pi_k(B))$ postcomposed with ξ gives a unique map $f' \in \pi_k(BO(m-k))$. This gives a twisted disk bundle X_f, with m dimensional total space. One can show that replacing $(M, \bar{\nu})$ with some $(M \# X_f, \bar{\nu} \# \nu_{X_f})$ kills the element f.
- Ker: This is a harder case. If $g \in \ker(\pi_{k+1}(M) \to \pi_{k+1}(B)$, then postcomposition $\xi \circ g : M \to BO$ is also zero and it gives the required bundle data to surger g out of M.

Determining the normal 1-type

There is the following theorem about 1-types:

Theorem 6 ([Tei92], pg. 8). *For a closed, smooth 4-manifold M, its 1-type is determined by $\pi := \pi_1(M), w_1(M), w_2(M), w_2(\tilde{M})$.*

Assume we are in the case where where the universal cover of M is Spin. The following Theorem asserts that for any 1-universal fibration $B \xrightarrow{\xi} \mathrm{BO}$, under an additional assumption which turns out that to be equivalent to $w_2(\tilde{M}) = 0$ because of the following exact sequence,

$$0 \to H^2(\pi; \mathbb{Z}/2) \to H^2(M; \mathbb{Z}/2) \to H^2(\tilde{M}, \mathbb{Z}/2)$$

there is a spectral sequence computing the bordism groups $\Omega_*(\xi)$:

Theorem 7 ([Tei93]). *For a universal 1-fibration $B \xrightarrow{\xi} \mathrm{BO}$, classes $w_1, w_2 \in H^*(B, \mathbb{Z}/2)$ in degree 1 and 2 respectively, which factor through $H^*(B\pi_1(B), \mathbb{Z}/2)$, there is a spectral sequence $H_p(B\pi_1(B); (\Omega_q^{\mathrm{Spin}})^{w_1}) \Longrightarrow \Omega_{p+q}(\xi)$.*

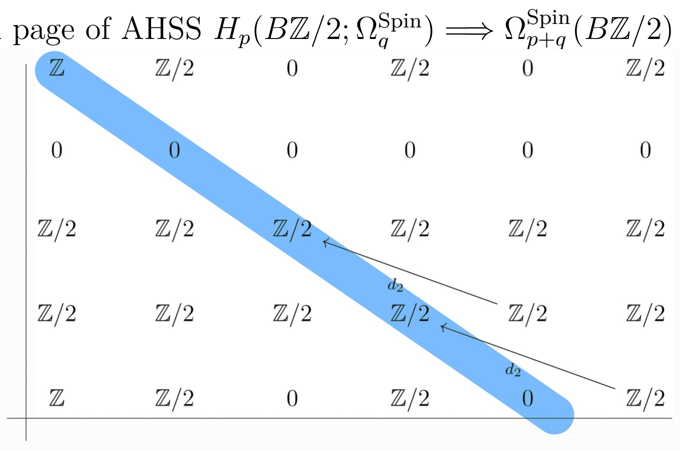
With the assumptions of the previous theorem it turns out that 1-type can be calculated by the following homotopy pullback.

$$\begin{array}{ccc} B & \longrightarrow & B\pi \\ \downarrow \xi & & \downarrow w_1 \times w_2 \\ BO & \xrightarrow[w_1(\gamma) \times w_2(\gamma)]{} & K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2) \end{array}$$

Fundamental group $\mathbb{Z}/2$

Example. Let us calculate the stable classification of orientable, spin 4-manifolds, with fundamental group $\mathbb{Z}/2$. The right vertical map of the square on the left is zero, so the homotopy pullback is $\mathrm{Hofib}_{w_1(\gamma) \times w_2(\gamma)} \times B\mathbb{Z}/2 \simeq BSpin \times B\mathbb{Z}/2$ and $\xi = i \circ p_1$. The corresponding bordism groups $\Omega_*(\xi)$ are in this case the ordinary bordism groups $\Omega_*^{\mathrm{Spin}}(B\mathbb{Z}/2)$ and can be calculated using the Atiyah–Hirzebruch spectral sequence. By Kreck's Theorem 4 $\Omega_4^{\mathrm{Spin}}(B\mathbb{Z}/2)$ is canonically in bijection with the stable diffeomorphism classes of manifolds and 1-smoothings.

The second page of AHSS $H_p(B\mathbb{Z}/2; \Omega_q^{\mathrm{Spin}}) \Longrightarrow \Omega_{p+q}^{\mathrm{Spin}}(B\mathbb{Z}/2)$ is



where we use homology of $B\mathbb{Z}/2 \simeq \mathbb{R}P^\infty$ and the spin bordism groups $\Omega_i^{\mathrm{Spin}} = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}$ for $i = 0, 1, 2, 3, 4$.

But the labeled differentials d_2 are known ([Tei92], Lemma 2.3.2)

- $d_2 = \mathrm{Sq}_2 \circ r_2 : H_5(\mathbb{R}P^\infty; \mathbb{Z}) \to H_3(\mathbb{R}P^\infty; \mathbb{Z}/2)$, the mod 2 reduction postcomposed with Sq_2 the dual of the Steenrod square.
- $d_2 = \mathrm{Sq}_2 : H_4(\mathbb{R}P^\infty; \mathbb{Z}/2) \to H_2(\mathbb{R}P^\infty; \mathbb{Z}/2)$

Both of these can be calculated to be isomorphisms in this case using the short exact sequence of coefficients $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ and properties of Steenrod squares $\mathrm{Sq}^2(x^2) = x^4$, $\mathrm{Sq}^2(x^3) = x^5$.

We find that $\Omega_4^{\mathrm{Spin}}(B\mathbb{Z}/2) \cong \Omega_4^{\mathrm{Spin}} \cong 16\mathbb{Z}$ where the last map is the signature. Two manifolds with a differing signature cannot be stably diffeomorphic(note the remark bellow), so we get:

Corollary 8. *Two oriented, spin, closed, 4-dimensional manifold with $\pi_1 = \mathbb{Z}/2$ are stably diffeomorphic if and only if their signature agrees.*

Remark. As with classical surgery where the structure set $\mathscr{S}_n(X)$ does NOT classify n-manifolds homotopy equivalent to X up to h,(resp. s)-cobordism, but manifolds with homotopy equivalence to $M \xrightarrow{f} X$ up to h,(resp. s)- cobordism with a map to X extending f, so in modified surgery $\mathrm{NSt}(\xi)$ classifies manifolds with 1-smoothing, up to stable diffeomorphism. In general we would have to divide out the various choices for the 1-smoothing. In this example this problem was luckily avoided.

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