UNIVERZITA KOMENSKÉHO V BRATISLAVE

FAKULTA MATEMATIKY, FYZIKY A INFORMATIKY

The Spivak Normal Fibration

Diploma Thesis

2021

Bc. Simona Veselá

UNIVERZITA KOMENSKÉHO V BRATISLAVE

FAKULTA MATEMATIKY, FYZIKY A INFORMATIKY

The Spivak Normal Fibration

Spivakova Normálová Fibrácia

Diploma Thesis

Študijný program:	Matematika
Študijný odbor:	Matematika
Školiace pracovisko:	FMFI.KAG - Katedra algebry a geometrie
Školiteľ:	Doc. Tibor Macko, PhD.

Bratislava, 2021

Bc. Simona Veselá

49841398

Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta:	Bc. Simona Veselá
Študijný program:	matematika (Jednoodborové štúdium, magisterský II. st., denná forma)
Študijný odbor:	matematika
Typ záverečnej práce:	diplomová
Jazyk záverečnej práce:	anglický
Sekundárny jazyk:	slovenský

Názov: The Spivak Normal Fibration
Spivakova normálová fibrácia

Anotácia: Je potrebné naštudovať a prezentovať definície a základne vlastnosti sférických fibrácií. Následne budú tieto aplikované na definíciu a dôkaz existencie Spivakovej normálovej fibrácie a Spivakovej normálovej štruktúry pre Poincarého priestory. Ďalšie dôležité témy sú klasifikácia sférických fibrácií, reducibilita Spivakovej normálovej fibrácie na vektorový bandl a príklad neexistencie takej redukcie vo všeobecnosti.

Cieľ:
1. Naštudovať a prezentovať definíciu a vlastnosti sférických fibrácií.
2. Naštudovať a prezentovať definíciu a vlastnosti Spivakovej normálovej štruktúry pre Poincarého priestory a dôkaz jej existencie pre jednoducho súvislé priestory.
3. Pokúsiť sa vylepšiť známe výsledky, prípadne vylepšiť prezentáciu z literatúry podrobnými vysvetleniami argumentov a ilustračnými príkladmi, alebo aspoň ilustrovať známe výsledky konrétnymi príkladmi.

Literatúra:

1. Allan Hatcher. Algebraic topology. Cambridge University Press; 1st edition, 2001.
2. Morris W. Hirsch. Differential Topology. Graduate Texts in Mathematics, Springer-Verlag, New York, 1976.
3. Ib Madsen and R. James Milgram. The classifying spaces for surgery and cobordism of manifolds, volume (AM-92). Princeton University Press, 1979
4. Andrew Ranicki. Algebraic and Geometric Surgery. Oxford Mathematical Monograph, Oxford, 2007.
5. Diarmuid Crowley, Wolfgang Lueck, Tibor Macko: Surgery theory - Foundations, book in preparation, 2020, available at https://www.mat.savba.sk/~macko/surgery-book.html

Kľúčové slová: vektorový bandl, sférická fibrácia, Spivakova normálová fibrácia, Spivakova normálová štruktúra, exotický Poincarého priestor

Vedúci:	doc. Mgr. Tibor Macko, PhD.
Katedra:	FMFI.KAG - Katedra algebry a geometrie
Vedúci katedry:	doc. RNDr. Pavel Chalmovianský, PhD.

Spôsob sprístupnenia elektronickej verzie práce:

49841398

Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

bez obmedzenia

Dátum zadania: 12.12.2019

Dátum schválenia: 12.12.2019

prof. RNDr. Ján Filo, CSc.
garant študijného programu

....
študent

....
vedúci práce

Týmto by som chcela poďakovať svojim rodičom za to, že mi veria vo všetkom, o čo sa snažím.
Taktiež chcem poďakovať môjmu školiteľovi Doc. Tiborovi Mackovi, Phd. za zaujímavú tému aj za ochotu a trpezlivosť.

I hereby declare that this thesis has been written by me with the use of the cited references and conversations with my supervisor.

Čestne prehlasujem, že som túto prácu vypracovala samostatne s použitím citovaných zdrojov a konzultácií s vedúcim práce.

Bratislava May 3, 2021

....

Abstract: The investigation of the Spivak normal fibration of a Poincaré complex gives an obstruction to a finite CW complex being homotopy equivalent to a manifold. In this thesis, we present the construction of the Spivak normal structure and demonstrate the non-vanishing of this obstruction for a well-known five-dimensional Poincaré complex X^5. In the literature this space is given by two different explicit constructions (5.27 and 5.33). We show that they are homotopy equivalent. We also consider finite simply connected CW complexes and we describe the result (Theorem 3.3.1) that such an X is a Poincaré complex if and only if the complement of X in its regular neighbourhood is up to homotopy a spherical fibration. In the positive case this yields the Spivak normal fibration of X.

Key words: vector bundle, spherical fibration, Spivak normal fibration, Spivak normal structure, exotic Poincaré complex

Abstrakt: Skúmanie Spivakovej normálnej fibrácie Poincarého priestoru dáva obštrukciu tomu, aby bol konečný CW priestor homotopicky ekvivalentný nejakej variete. V tejto práci uvedieme konštrukciu Spivakovej normálnej štruktúry a ukážeme nezaniknutie tejto obštrukcie pre jeden známy päťrozmerný Poincarého priestor X^5. V literatúre je tento priestor často daný dvoma rôznymi konštrukciami (5.27 a 5.33). Ukážeme, že sú homotopicky ekvivalentné. Ďalej uvažujeme konečné jednoducho súvislé CW priestory a uvedieme výsledok (Veta 3.3.1), že taký priestor X je Poincarého priestor vtedy a len vtedy, ak doplnok X jeho regulárneho okolia je až na homotópiu sférická fibrácia. Ak je táto podmienka splnená, takáto konštrukcia nám dáva Spivakovu normálovú fibráciu priestoru X.

Kľúčové slová: vektorový bandl, sférická fibrácia, Spivakova normálová fibrácia, Spivakova normálová štruktúra, exotický Poincarého priestor

Table of Contents

Introduction

Surgery Theory is a branch of Algebraic Topology that develops methods to classify manifolds of dimensions ≥ 5. It began with the publication of the paper *Groups of Homotopy Spheres* in the year 1963 by Michel A. Kervaire and John W. Milnor concerning the classification of exotic spheres [12, Preface].

An important question in surgery theory is deciding whether a topological space is homotopy equivalent to a compact manifold. Manifolds in question may be topological manifolds, smooth manifolds or other types, but in this thesis, we only consider smooth manifolds. Since every smooth compact manifold has the homotopy type of a finite CW complex *, we may restrict our attention to finite CW complexes only.

In the first Chapter of this thesis, we summarise some topological concepts which are needed later. In Section 1.6 we recall the Poincaré duality and define Poincaré complexes as finite CW complexes which have Poincaré duality. From now on we only consider simply connected finite CW complexes. Otherwise, the theory gets more complicated with the introduction of local coefficients.

In the second Chapter we define the spherical fibrations, that means fibrations with the fiber homotopy equivalent to a sphere (Section 2.1). In Section 2.3 we state the homological and cohomological version of the Thom isomorphism Theorem for spherical fibrations. We prove the Theorem in detail for the trivial fibrations which is the case usually left as an exercise in the proofs in the literature.

The third Chapter is on the Spivak normal structure, due to M. Spivak [14], part of which is the Spivak normal fibration. It is a spherical fibration over a Poincaré complex. It serves as an analogue of the normal vector bundle of a manifold. Specifically, it gives another obstruction to the Poincaré complex being homotopy equivalent to a manifold. Namely, there is no such homotopy equivalence when the Spivak normal fibration does not have a *vector bundle reduction*. Roughly speaking, a vector bundle reduction of a Spivak normal fibration is an indication that this fibration could have originated from a normal bundle of some manifold. This is explained in Section 5.1 in Chapter 5.

*This is a consequence of Morse theory. See e.g. [1, Section 1.3]

A candidate for the Spivak normal fibration of a finite CW complex X is obtained by taking the complement of its regular neighbourhood after embedding X into a large Euclidean space. It is up to homotopy a spherical fibration if and only if X is a Poincare complex. We show it for simply connected finite CW complexes (Theorem 3.3.1). This is done with the Thom isomorphism Theorem (2.3.5) and its converse (2.3.6), which show that a fibration is spherical if and only if the Thom homomorphism is an isomorphism. In that case, we get an equation $U_p \frown h(c) = [X]$, which connects the Thom class U_p of the Spivak normal fibration and the fundamental class $[X]$ of X (the element $h(c)$ is given as part of the Spivak normal structure).

In the fourth Chapter we describe the classification results for both vector bundles and spherical fibration, the latter being due to J. D. Stacheff [15]. There are classifying spaces $BO(k)$, BO of k-dimensional vector spaces and stable vector bundles as well as $BG(k)$, BG the classifying spaces of $(k-1)$-dimensional spherical fibrations and stable spherical fibrations. Knowing the homotopy groups of these spaces facilitates the example of the nonexistence of the vector bundle reduction of the Spivak normal fibration for a particular space in the next chapter.

In the last Chapter we see this well-known 5-dimensional Poincaré space X^5 which is not homotopy equivalent to a manifold. In the literature it is defined as either two copies of $S^2 \times D^3$ glued together along the boundary $S^2 \times \partial D^3$ by a particular homotopy equivalence (see 5.27) or it is defined as $D^5 \cup_f S^2 \vee S^3$ for a certain map $f \in \pi_4(S^2 \vee S^3)$ (see 5.33). In the rest of the chapter, we prove, in detail, that these two definitions yield homotopy equivalent spaces.

Chapter 1

Some Topological Prerequisites

The convention in this thesis is that all homologies, all cohomologies, all tensor products are with integral coefficients ($\mathbb{Z}$). In all proofs where it is relevant, we use cellular (co)homology. All our manifolds are smooth, compact manifolds, all maps between manifolds are smooth. All maps between topological spaces are continuous. All vector bundles are real vector bundles unless stated otherwise.

We rely heavily on the unfinished book *Surgery Theory: Foundations* by Diarmuid Crowley, Wolfgang Lück and Tibor Macko [1]. This makes citing from it a little difficult. We rely solely on the version from 6. December 2020, which can be accessed from the link in the Bibliography.

1.1 CW complexes

Let us take a topological space X. Now we define the process of *attaching an n-cell*. Let us have a map $\varphi : S^{n-1} \to X$. We define $X \cup_\varphi D^n$ as the following pushout diagram.

$$\begin{tikzcd} S^{n-1} \arrow[r, "\varphi"] \arrow[d, hook] & X \arrow[d] \\ D^n \arrow[r, "\phi"] & X \cup_\varphi D^n \end{tikzcd} \tag{1.1}$$

Using the typical inclusion $S^{n-1} \hookrightarrow D^n$. Intuitively $X \cup_\varphi D^n$ is the space X with an n-disk glued by its boundary to X in a way defined by the map φ.

We call φ the *attaching map* and ϕ the *characteristic map.*

Definition 1.1.1. [5, p. 5] A finite n-dimensional CW complex is a topological space X together with a filtration $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subsetneq X_n = X$. Such that

X_k, $n \geq k \geq 0$, a so-called k-skeleton is formed from X_{k-1} by attaching a finite number of n-cells. For $k > n$ we define a k-skeleton as $X_k = X_n$.

There are also non-finite CW complexes, which we will not need in this thesis.

Definition 1.1.2. [5, p. 7] A subcomplex A of a CW complex X is a CW complex which is a union of some cells of X. In this case we may call (X, A) a *CW pair*.

Definition 1.1.3. Lat us have a map of $f : X \to Y$ of CW complexes. Then f is called cellular if f preserves the structure of skeletons, if we have $f(X_k) \subset Y_k$ for all k.

Definition 1.1.4. [5, Theorem A.5] Let X and Y be two finite CW complexes of dimensions n and m respectively. Denote the characteristic maps of the cells in X by Φ^i_α and for cells in Y by Ψ^j_β. Then their product topology has a natural structure of a nm-dimensional finite CW complex with $0 \leq k \leq nm$ cells having characteristic maps $\Phi^i_\alpha \times \Psi^j_\beta : D^i \times D^j \to X \times Y$ for all i, j such that $i + j = k$.

Definition 1.1.5. [5, p. 346] and [12, Definition 3.7, p. 31] A map $f : X \to Y$ is n-connected if one of the equivalent conditions is satisfied.

1. For all $x \in X$ the induced morphism $\pi_i(X, x) \xrightarrow{f_*} \pi_i(Y, f(x))$ is an
   - Isomorphism for $0 < i < n$ (where the isomorphism of the zeroth homotopy is a bijection).
   - Epimorphism for $i = n$ (where the epimorphism of the zeroth homotopy is a surjection).
2. For all $0 \leq i \leq n$ and for all commutative squares

$$\begin{tikzcd} S^{i-1} \arrow[r, "u"] \arrow[d] & X \arrow[d, "f"] \\ D^i \arrow[r, "v"] & Y \end{tikzcd} \tag{1.2}$$

   There is a map $w : D^i \to X$ such that $w \restriction S^{i-1} = u$ and $f \circ w$ is homotopic to v relative S^{i-1}.

A space X is n-connected $\pi_i(X, x)$ is trivial for all $i \leq n$, $x \in X$.

A pair of spaces (X, A) is n-connected if the inclusion map $i : A \to X$ is n-connected. In this case, this is equivalent to a third condition describing relative homotopy groups

3. $\pi_i(X, A, x) = 0$ for $n \geq i > 0$ and any path component of X contains a point from A (the second part replaces the concept of the zeroth relative homotopy groups).

Proposition 1.1.6. *(Excision Theorem for homotopy groups)[5, Prop. 4.28, p. 364] Let (X, A) be an r-connected pair and A be s-connected for some $r, s \geq 0$. Then the map $\pi_i(X, A) \to \pi_i(X/A)$ induced by the quotient map is an isomorphism for $i \leq r + s$ and is surjective for $i = r + s + 1$.*

1.2 Basic Constructions

Definition 1.2.1. Let X, Y be spaces. Denote by $[X, Y]$ the homotopy classes of unpointed maps from X to Y.

Note that the homotopy in the previous definition is a free homotopy. For example in homotopy groups, homotopy classes relative basepoints of S^n are used.

Definition 1.2.2. Let X, Y be spaces. Denote by $\langle X, Y \rangle$ the homotopy classes of pointed maps from X to Y. Here homotopy is taken to be relative the basepoint.

Definition 1.2.3. (The Wedge Sum) Let X and Y be two pointed topological spaces with basepoints $x_0 \in X$ and $y_0 \in Y$. The *wedge sum* of X and Y is a pointed topological space where X and Y are glued by their basepoints $X \vee Y = X \cup_{x_0 \sim y_0} Y$. The basepoint of the wedge sum is the point $x_0 \sim y_0$.

For two maps $f : X_1 \to Y$, $g : X_2 \to Y$ of pointed topological spaces define the wedge of maps as

$$\begin{aligned} f \vee g : & X_1 \vee X_2 \to Y \\ & x \mapsto f(x) \quad x \in X_1 \\ & x \mapsto g(x) \quad x \in X_2 \end{aligned} \tag{1.3}$$

This map is continuous and well defined in the basepoint of $X_1 \vee X_2$.

Definition 1.2.4. (Unreduced Suspension) Let X be an unpointed topological space. Define its (unreduced) suspension SX as $X \times [0, 1]/X \times \{0\} \coprod X \times \{1\}$.

Definition 1.2.5. (Reduced Suspension) Let X be an pointed topological space. Define the reduced suspension ΣX as $X \times [0, 1]/(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times [0, 1])$. We can canonically chose a basepoint of ΣX as the point
$(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times [0, 1]) \,/\, (X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times [0, 1])$.

Let $f : X \to Y$ by a map of spaces. For both suspensions, we also have suspensions of maps

$$Sf : SX \to SY \tag{1.4}$$

$$\Sigma f : \Sigma X \to \Sigma Y \tag{1.5}$$

This distinction between these suspensions does not always matter since we have

Lemma 1.2.6. *For a pointed CW complex, X the unreduced suspension SX is homotopy equivalent to the reduced suspensions ΣX .*

The proof follows from the fact that $(SX, \{x_0\} \times [0,1])$ is a CW pair, $\{x_0\} \times [0,1]$ is contractible and hence $SX \xrightarrow{\simeq} SX/(\{x_0\} \times [0,1]) = \Sigma X$ by [18, Theorem 5.13, p.26].

For example, (reduced or unreduced) suspension of an n-sphere is homotopy equivalent to an $(n+1)$-sphere.

A suspension gives the following homomorphism of homotopy groups.

Definition 1.2.7. For a pointed space X and an integer $k \geq 0$ define the *suspension map* $\sigma : \pi_k(X) \to \pi_{k+1}(\Sigma X)$ by sending a pointed map $f : S^k \to X$ to the pointed map $\Sigma f : \Sigma S^k \simeq S^{k+1} \to \Sigma X$.

Definition 1.2.8. [5, p. 10] Let X and Y be pointed spaces. Then there is a pointed analogue of the Cartesian product of spaces, namely the *smash product*. Define

$$X \wedge Y = X \times Y / X \vee Y \tag{1.6}$$

where the inclusion $X \vee Y \to X \times Y$ is defined by $X \mapsto X \times \{y_0\}$ and $Y \mapsto \{x_0\} \times Y$. The basepoint of the smash product is $X \vee Y / X \vee Y$.

1.3 Nice Neighbourhoods

Definition 1.3.1. Let $r : X \to A$ be a a map of spaces. It is a *retraction* if there exists a map $i : A \to X$ such that $r \circ i = \mathrm{id}_A$. We can also say that r is a *retraction* of i. In this case the map i is called a *section* of r.

If A is a subspace of X, then A is called a *retract* of X if there is a retraction $r : X \to A$ of the inclusion $i : A \to X$.

A stronger condition is for A to be a *strong deformation retract*. It is when for an inclusion $i : A \to X$ there is a map $r : X \times I \to X$ which is a homotopy relative

$i(A)$ of $r_0 = \mathrm{id}_X$ and $r_1 : X \to A$ a retraction of i. The map r is then called a *strong deformation retraction.*

Note that this terminology varies throughout the literature.

Definition 1.3.2. [3, Definition 1.2, p. 141] A map $i : A \to X$ of topological spaces is a *(Hurewicz) cofibration* if it satisfies the following homotopy extension property: For any space Y, any map $g_0 : X \times \{0\} \to Y$ and a homotopy $g : A \times I \to Y$ in the diagram

$$\begin{tikzcd}
A \times \{0\} \arrow[rr] \arrow[dd, "f\times\mathrm{id}"] & & A \times I \arrow[dl, "g"'] \arrow[dd, "f\times\mathrm{id}"] \\
& Y & \\
X \times \{0\} \arrow[ur, "g_0"] \arrow[rr] & & X \times I \arrow[ul, dotted, "\widetilde{g}"]
\end{tikzcd} \tag{1.7}$$

there exists a map $\widetilde{g} : X \times I \to Y$ which fits into the diagram.

Example 1.3.3. For a CW pair (X, A), the inclusion $i : A \to X$ is a cofibration[3, Corollary 1.4, p. 431].

Any finite CW complex is embeddable in a sufficiently large Euclidean space. This is shown for example in the proof of the Corollary A.10 in [5, p. 527]. Any 0-skeleton of a finite CW complex is embeddable in $\mathbb{R}$. Let us assume that we have an embedding of $A \to \mathbb{R}^m$. Assume we attach to A a k-cell by the map φ. Then $A \cup_\varphi D^k$ is embeddable in $\mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^1$ as union of $(D^k \times \{0\} \times \{0\})$, $(\{0\} \times A \times \{1\})$ and line segments joining $(x, 0, 0)$ with $(0, \varphi(x), 1)$ for all $x \in \partial D^n$. This only takes finite number of steps.

This construction is rather wasteful when it comes to dimensions. Actually an n-dimensional CW complex can always be embedded in a $(2n+1)$-dimensional Euclidean space [13, Chapter 3].

Definition 1.3.4. [1, p. 158] Let X be a finite n-dimensional CW complex, embedded in a Euclidean space $i_X : X \to \mathbb{R}^{n+k}$. Then a regular neighbourhood of X in $\mathbb{R}^{n+k}$ is a compact manifold N with boundary ∂N such that N is a neighbourhood of $i_X(X)$ and there is a strong deformation retract $r : N \times I \to N$ of the inclusion i_X.

There is the following uniqueness result:

Theorem 1.3.5. *[13, Theorem 3.24, p. 38] For a finite n-dimensional CW complex X embedded in a Euclidean space $i_X : X \to \mathbb{R}^{n+k}$ and its two regular neighbourhoods $(N_1, \partial N_1)$, $(N_2, \partial N_2)$ there is an isotopy with compact support $V \times I \to V$ such that H is stationary on X, $H_0 = \mathrm{id}_V$ and $H_1(N_1) = N_2$.*

Whitney embedding theorems (e.g. [3, Theorem 10.7, p. 91]) gives us an embedding of any compact manifold to a sufficiently large Euclidean space.

Definition 1.3.6. [7, Section 5, p. 109] Let $i : M \to \mathbb{R}^{n+k}$ be an embedding of an n-dimensional manifold M without a boundary into a Euclidean space $\mathbb{R}^{n+k}$. A *tubular neighbourhood* of this embedding is a pair (f, ν) such that $\nu : E \to M$ is a vector bundle with M identified with its zero section and an embedding $f : E \to \mathbb{R}^{n+k}$ such that

- $f \restriction M = \mathrm{id}_M$.
- $f(E)$ is an open neighbourhood of M in $\mathbb{R}^{n+k}$.

The existence of such neighbourhood is proved, for example in [7, Theorem 5.1, p. 109]

The tubular neighbourhood of an embedding is unique up to isotopy [7, Theorem 5.3, p. 109].

The definition of a tubular neighbourhood (and hence of a normal bundle of a manifold) depends on the chosen embedding into a Euclidean space. There is a way in which this choice does not matter, this will be explained in the Note 2.2.2.

There is a sense in which the regular neighbourhood is a generalisation of the tubular neighbourhood. Let (f, ν) be a tubular neighbourhood (f, ν) of a manifold M, then the associated disk and sphere bundles $(D\nu, S\nu)$ (see Definition 2.1.20) of ν form the regular neighbourhood of M with the strong deformation retract being shortening of the vectors in $D\nu$.

1.4 Colimits and Homotopy Colimits

Colimit is a construction in category theory. Homotopy colimit represents a variant of this notion in a so-called homotopy category. To rigorously define these concepts is beyond the scope of this thesis. We will require to take so-called *sequential colimits/homotopy colimits* in the category of groups and topological spaces. These are colimits of a sequence like this:

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \tag{1.8}$$

The sequential colimit or a homotopy colimit in a particular category is an object in the same category. Denote this colimit and homotopy colimit by

$$X = \mathrm{colim}_{k \to \infty}(X_k) \tag{1.9}$$

and

$$X = \operatorname{hocolim}_{k\to\infty}(X_k) \tag{1.10}$$

respectively.

We can describe the sequential colimit in the category of topological spaces. The colimit is the set $X = \bigcup X_k/\sim$ such that two points are identified by $\sim$ if they are eventually identified in some X_i by successive application of the functions f_j. This set is endowed with the *final topology*, i.e. the largest topology such that the obvious maps $X_i \to X$ are continuous.

This construction is not homotopy invariant. More precisely, when we take colimit of sequence 1.8, we may not be able to exchange objects for homotopy equivalent ones, nor maps for homotopic ones. This is the reason for homotopy colimits, where those operations are allowed, in other words, exchanging object and maps up to homotopy yields the same result up to homotopy.

Some useful properties of colimits (of groups and spaces) and homotopy colimits (of spaces):

Lemma 1.4.1. *(Properties of Colimits and Homotopy Colimits)[17, 6.3.2, p. 152 and 6.23, p.152 and Theorem 6.27, p. 154]*

i) For sequential (homotopy) colimits 1.8 there are natural maps $X_i \to \operatorname{colim}_{k\to\infty}(X_k)$ and $X_i \to \operatorname{hocolim}_{k\to\infty}(X_k)$ for each $i \geq 0$.

ii) Let $\{X_k\}_{0\leq k}$ and $\{Y_k\}_{0\leq k}$ be two sequences as in 1.8, X and Y their colimits (or homotopy colimits) assume we have maps between the terms $X_k \xrightarrow{g_k} Y_k$. Then there is a natural map $X \xrightarrow{g} Y$ such that the following diagram commutes for any k

$$\begin{tikzcd} X_k \arrow[r, "g_k"] \arrow[d] & Y_k \arrow[d] \\ X \arrow[r, "g"] & Y \end{tikzcd} \tag{1.11}$$

iii) In the case of sequential colimits of groups, sequential colimits of topological spaces, sequential homotopy colimits of topological spaces if the maps g_k from ii) are isomorphisms, homeomorphisms, homotopy equivalence respectively so is the map g.

iv) *Sequential colimits and homotopy colimits have the property that that any finite number of terms from the beginning can be discarded and the resulting colimit or homotopy colimit will be isomorphic or homotopy equivalent respectively.*

v) *[1, 5.26, p. 145] Let us have a sequence 1.8 of topological spaces. Then there is an isomorphism*

$$\operatorname{colim}_{k\to\infty} \pi_k(X_k) \xrightarrow{\cong} \pi_k(\operatorname{hocolim}_{k\to\infty} X_k) \tag{1.12}$$

There is a useful result, which indicates when it is safe to work with ordinary colimits and still change the terms and maps up to homotopy.

Lemma 1.4.2. *[1, p. 145] Let $\{X_k\}_{0\leq k}$ be the sequence of topological spaces as in 1.8, where each $f_k : X_k \to X_{k+1}$ is a cofibration. Then*

i) *[17, pp. 154-155] There is a homotopy equivalence*

$$\operatorname{colim}_{k\to\infty} X_k \xrightarrow{\simeq} \operatorname{hocolim}_{k\to\infty} X_k \tag{1.13}$$

ii) *For homotopy groups we have* $\operatorname{colim}_{k\to\infty} \pi_i(X_k) \xrightarrow{\cong} \pi_i(\operatorname{colim}_{k\to\infty} X_k)$.

1.5 Some Homotopy Groups of Spheres

The reference for this section is [5, Chapter 4, 384ff].

Recall that we had the suspension homomorphism 1.2.7 of homotopy groups. Successive application of this homomorphism for $k \geq 0$ yields a sequence

$$\pi_k(S^0) \xrightarrow{\ \sigma\ } \pi_{k+1}(S^1) \xrightarrow{\ \sigma\ } \pi_{k+2}(S^2) \xrightarrow{\ \sigma\ } \pi_{k+3}(S^3) \xrightarrow{\ \sigma\ } \cdots \tag{1.14}$$

The colimit of such sequence is called the *stable stem* and denoted by π_k^s. The sequence always stabilises by the Freudenthal suspension theorem. In particular for $n \geq k+2$ the group $\pi_{n+k}(S^n)$ is already stable, though this may happen sooner (i.e. for smaller n).

We will be interested in the groups of the sequence with $k = 1$ and $k = 2$.

Let us start with $k = 1$. The groups $\pi_1(S^0)$, $\pi_2(S^1)$ are trivial, while the group $\pi_3(S^2)$ is isomorphic to $\mathbb{Z}$, its generator is $\eta : S^3 \to S^2$ the Hopf fibration. The group stabilisation $\pi_3(S^2) \xrightarrow{\sigma} \pi_4(S^3)$ is injective, the group $\pi_4(S^3)$ is $\mathbb{Z}/2$ and is already stable.

For $k = 2$, the groups $\pi_2(S^0)$, $\pi_3(S^1)$ are trivial, while the group $\pi_4(S^2)$ is isomorphic to $\mathbb{Z}/2$ and is already stable.

This can be deduced if we know the Hopf fibration $S^1 \to S^3 \xrightarrow{\eta} S^2$. The long exact sequence of this fibration gives

$$\pi_4(S^1) \longrightarrow \pi_4(S^3) \xrightarrow{\eta_*} \pi_4(S^2) \longrightarrow \pi_3(S^1) \tag{1.15}$$

The triviality of $\pi_k(S^1)$ for $k > 1$ can be shown from the universal covering space of a circle $\mathbb{R} \to S^1$. The higher homotopy groups of a space and its covering space coincide [5, prop 4.1, p. 342]. The space $\mathbb{R}$ is contractible.

From 1.15 we get an isomorphism $\eta_* : \mathbb{Z}/2 \cong \pi_4(S^3) \xrightarrow{\eta_*} \pi_4(S^2)$. As we have seen, the suspension $\sigma\eta$ is the generator of $\pi_4(S^3)$. So the element $\eta_*(\sigma\eta)$ is the generator of $\pi_4(S^2)$. But maps induced in homotopy groups are postcomposition, so we have $\eta_*(\sigma\eta) = \eta \circ \sigma\eta$ the generator of $\pi_4(S^2)$.

1.6 Poincaré Duality

Here we will summarise results about Poincaré duality for simply connected manifolds. The Poincaré duality in this chapter only holds for oriented manifolds. We will define the notion of orientability and then show that simply connected manifolds are orientable, hence for simply connected CW complexes, this Poincaré duality is sufficient.

This section follows the book of Allan Hatcher [5].

There are various equivalent ways how to define orientability for manifolds. One can either chose orientation for the tangent space T_xM at each point x of the manifold in a way that is locally consistent. The following is a suitable definition for this purpose.

Definition 1.6.1. [5, p. 233]

Let M be an n-dimensional manifold and a point in it $x \in M$.

Then a *local orientation* at x is a choice of the generator μ_x of the infinite cyclic group $H_n(M, M \setminus \{x\})$.

An *orientation* of M is a function $x \mapsto \mu_x$ assigning to each point on M a choice of local orientation such that the choice is consistent in the following sense:

For each point $x \in M$ with a local trivialisation $x \in U \xrightarrow{\cong} \mathbb{R}^n$ there exists an open ball $B \subset U$ such that for each $y \in B$ both μ_x and μ_y are images of the same generator of $H_n(M, M \setminus B)$ under the obvious maps

$$H_n(M, M \setminus B) \to H_n(M, M \setminus \{x\})$$

$$H_n(M, M \setminus B) \to H_n(M, M \setminus \{y\})$$

A manifold M is called *orientable* if there is such an orientation function.

A manifold with a boundary $(M, \partial M)$ is said to be orientable if its interior $M \setminus \partial M$ is an orientable manifold.

There are a few claims in this definition which we shall unravel. Firstly we need to show that for a contractible open subset B of M we have that $H_n(M, M \setminus B)$ is an infinite cyclic group

$$H_n(M, M \setminus B) \cong H_n(B, \partial B) \cong H_n(D^n, S^{n-1}) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}$$

where the first isomorphism is by excision.

From this, we also get that for a point $x \in M$ and some neighbourhood B homeomorphic to $\mathbb{R}^n$ we get

$$H_n(M, M \setminus \{x\}) \cong H_n(B, B \setminus \{x\}) \cong \mathbb{Z}$$

Lemma 1.6.2. *[5, p. 234] For any manifold M there is a two-sheeted covering space $\widetilde{M} \xrightarrow{p} M$ called a double cover or an orientation double cover, where $\widetilde{M} = \{\mu_x | \; x \in M, \mu_x \text{ is a local orientation}\}$ and*

$$\begin{aligned} p : \widetilde{M} &\to M \\ \mu_x &\mapsto x \end{aligned} \tag{1.16}$$

An *n-sheeted* covering space is a covering space where the preimage of a point consists of n discrete points.

The orientation double cover is defined for any manifold without any choices.

Proposition 1.6.3. *[5, Prop 3.25, p. 234] Let M be a connected manifold. Then M is orientable if and only if the double cover is disconnected.*

Since the covering space is 2-sheeted, it either has one component or two. In the latter case, we have $\widetilde{M} \cong M \coprod M \xrightarrow{p} M$ such that p is a homeomorphism on each component.

From the theory of covering spaces, it follows that for any simply connected space X, its only covering space which is connected is X itself. From this, it follows

Proposition 1.6.4. *Simply connected manifolds are orientable.*

Proposition 1.6.5. *(Fundamental class)[5, p. 236] Let M be an orientable n-dimensional manifold. Then there exists a generator, a so-called fundamental class $[M] \in H_n(M)$ which gets mapped to the local orientation at x under the map $H_n(M) \to H_n(M, M \setminus \{x\})$ for all x.*

Let X be a space and A, B open, possibly empty subspaces. Then there is the cup product ($\smile$) and the cap ($\frown$) product

$$H^k(X, A) \times H^l(X, B) \xrightarrow{\smile} H^{k+l}(X, A \cup B) \tag{1.17}$$

$$H^k(X, A) \times H_l(X, A \cup B) \xrightarrow{\frown} H_{k-l}(X, B) \tag{1.18}$$

Note that A and/or B can be empty. If they are both empty, this yields the cup/cap product in absolute (co)homologies.,

Note that the order of the groups in the cap product varies throughout the literature.

Theorem 1.6.6. *(Poincaré duality)[5, Theorem 3.30, p. 241] If M is a closed, compact, oriented n-dimensional manifold with a fundamental class $[M]$, then the following map is an isomorphism*

$$\begin{aligned} H^*(M) &\to H_{n-*}(M) \\ a &\mapsto a \frown [M] \end{aligned} \tag{1.19}$$

There is also a relative version of Poincaré duality. For an n-dimensional, compact manifold M with boundary ∂M the group $H_n(M, \partial M)$ is infinite cyclic and there is a generator $[M, \partial M]$ the fundamental class compatible with the orientation such that the following are isomorphisms (for the full construction and proof see e.g.[12, pp. 355-358])

$$- \frown [M, \partial M] : H^*(M, \partial M) \to H_{n-*}(M) \tag{1.20}$$

$$- \frown [M, \partial M] : H^*(M) \to H_{n-*}(M, \partial M) \tag{1.21}$$

It turns out that a space satisfying Poincaré duality is not necessarily homotopy equivalent to a manifold. An example of such space is the topic of the Chapter 5. This prompts the following definition

Definition 1.6.7. [1, Definition 4.43, p. 117] Let X be a finite, simply connected CW complex, n an integer. We call X a *finite, n-dimensional Poincaré complex* if there is an element $[X] \in H_n(X)$ such that the following map is an isomorphism

$$- \frown [X] : H^{n-*}(X) \to H_*(X) \tag{1.22}$$

The element $[X]$ from the previous definition is also called the *fundamental class* of X.

Chapter 2

Spherical Fibrations

2.1 Fibrations

Recall fiber bundles and vector bundles. Both are maps $p : E \to X$ satisfying some local triviality conditions. The preimage of a point $p^{-1}(x)$ is called a fiber and its homeomorphism type does not depend on x. Vector bundles have vector spaces as fibers and are also required to satisfy certain conditions which preserve their linear structure.

For homotopy theory, it is beneficial to work in a more general context which, for instance, requires that the preimage $p^{-1}(x)$ of a point be only homotopy equivalent to the fiber.

Definition 2.1.1. [5, p. 375] Let $p : E \to X$ be any map of spaces. We say that the map p satisfies the *homotopy lifting property* with respect to some space Y if for any homotopy $g : Y \times I \to X$ and a map $\widetilde{g_0} : Y \to E$ such that $p\widetilde{g_0} = g(0)$ we have a lift $\widetilde{g} : Y \times I \to E$ of g or in other words, the map $\widetilde{g}$ satisfies $p\widetilde{g} = g$.

This can be summarised in the following diagram:

$$\begin{tikzcd} Y \arrow[r, "\widetilde{g_0}"] \arrow[d, hook, "Y\times\{0\}"'] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "g"] \arrow[ur, dashed, "\widetilde{g}"] & X \end{tikzcd} \tag{2.1}$$

Definition 2.1.2. [5, p. 375] A map $p : X \to E$ is a so-called *Hurewicz* fibration if it has the homotopy lifting property with respect to any space.

As for the nomenclature, in both fiber bundles and fibrations, X is called the base space, E is called the total space and the preimage $p^{-1}(x_0) = F$ of a basepoint $x_0 \in X$ is called the fiber.

In this thesis, by fibration, we mean a Hurewicz fibration with the base space, total space and the fiber having the homotopy type of a finite CW complex.

If we want to stress the fiber F of a fibration $p : E \to X$ it is customary to write

$$F \to E \xrightarrow{p} X.$$

For $x \in X$, a point in the base space, we can denote E_x to be $p^{-1}(x)$, the preimage of x.

From the definition of fibrations, it is not yet clear that fibrations have the property that all fibers are homotopy equivalent, but this is indeed so:

Proposition 2.1.3. *[5, Prop. 4.61] For a fibration $p : E \to X$ the fibers $E_x = p^{-1}(x)$ for all points $x \in X$ are homotopy equivalent.*

The notion of a fibration is a generalisation of the notion of a fiber bundle (in the case of limiting our attention to CW complexes). To show this we use the Proposition 4.48 in [5, p. 379] which states that a fiber bundle has the homotopy lifting property with respect to any CW pair. According to the discussion on p. 376. in [5] this is equivalent to being a so-called *Serre fibration* which is defined as a map with homotopy lifting property with respect to all disks. In the case of CW complexes, this is equivalent to being a Hurewicz fibration [16, Theorem 1.]. So we have

Proposition 2.1.4. *All fiber bundles over CW complexes are fibrations.*

The converse is not true. The basic well-known example of this is a filled triangle with the projection onto the base, see Figure 2.1. This cannot be a fiber bundle since not all fibers are homeomorphic.

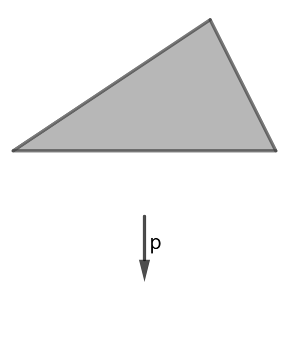

Figure 2.1: A fibration with fibers of different homeomorphism type

We will need the following important property of fibrations in later chapters.

Lemma 2.1.5. *[2, Theorem 6.29, p. 135]Let $F \xrightarrow{i} E \xrightarrow{p} X$ be a fibration with X path connected. Let Y be any space. Then the induced maps*

$$[Y, F] \xrightarrow{i_*} [Y, E] \xrightarrow{p_*} [Y, B]$$

are exact at $[Y, E]$, in other words $\mathrm{Im}(i_*) = \mathrm{Ker}(p_*)$.

From this Lemma there follows an important obstruction to lifting a map $f : Y \to E$ to a map $h : Y \to F$, such that $i \circ h = f$ as in the diagram 2.2.

$$\begin{tikzcd} F \arrow[r, "i"] & E \arrow[r, "p"] & X \\ & Y \arrow[u, "f"] \arrow[ul, dotted, "h"] & \end{tikzcd} \tag{2.2}$$

Let us form the claim in a form of a Lemma

Lemma 2.1.6. *The lift $h : Y \to F$ of $f : Y \to E$ in the diagram 2.2 exists if and only if the composition $p \circ f$ is null-homotopic ($p \circ f \simeq *$).*

Proof. Assume such lift exists. Then we have $i \circ h = f$ or f is in the image of the map $i_* : [Y, F] \to [Y, E]$. This is equivalent by the Lemma 2.1.5 to $f \in \mathrm{Ker}(p_*)$ for the map $p_* : [Y, E] \to [Y, X]$. This means that $p \circ f \simeq *$.

All the steps were equivalences so the proof is finished. □

Definition 2.1.7. Let X and F be any spaces. Then there is the following fibration and also a fiber bundle with fiber F called the *trivial fibration* or *trivial fiber bundle*

$$X \times F \to X \tag{2.3}$$
$$(x, z) \mapsto x$$

It is a fibration since for any Y, $\widetilde{g_0}$, g in the following homotopy lifting property diagram

$$\begin{tikzcd} Y \arrow[r, "\widetilde{g_0}"] \arrow[d, hook, "Y \times \{0\}"'] & X \times F \arrow[d, "p"] \\ Y \times I \arrow[r, "g"] \arrow[ur, dashed, "\widetilde{g}"] & X \end{tikzcd} \tag{2.4}$$

we define $\widetilde{g}$ by sending (y, t) to $(g(y, t), p_2 \circ \widetilde{g_0}(y))$, where p_2 is the projection onto the second coordinate $p_2 : X \times F \to F$.

If X is clear from the context we write the trivial fibration with fiber F as $\underline{F}$. This will be exclusively used for fibrations with fiber S^{k-1} as $\underline{S^{k-1}}$. Similarly, the trivial k-dimensional vector bundle will be denoted by $\underline{\mathbb{R}^k}$.

Definition 2.1.8. [5, p. 406] Let $p : E \to X$ be a fibration and $f : Y \to X$ any map of spaces . A *pullback* of p along f is a fibration $q : f^*(E) \to Y$ with $f^*(E) = \{(e, y)|\ e \in E, y \in Y, p(e) = f(y)\}$ and $q(e, y) = y$. There is also the map of the total spaces given by $(e, y) \mapsto e$.

$$\begin{tikzcd} f^*(E) \arrow[r, dashed] \arrow[d, dashed, "q"] & E \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd} \tag{2.5}$$

For the proof that q is fibration see [5, p. 406]. The fibration q has the same fiber as p, since $q^{-1}(y_0) = p^{-1}(f(y_0))$.

Now we define the main object of interest in this thesis.

Definition 2.1.9. Let $0 < k$. By a $(k-1)$-spherical fibration $p : E \to X$ we mean a fibration with fiber homotopy equivalent to S^{k-1}.

Now we define maps and homotopies between spherical fibrations

Definition 2.1.10. [1, p. 136] Let $p_0 : E_0 \to X_0$ and $p_1 : E_1 \to X_1$ be two spherical fibrations.

A *fiber map* $(f, \bar{f}) : p_0 \to p_1$ consists of maps $f : X_0 \to X_1$ and $\bar{f} : E_0 \to E_1$ such that the following diagram commutes

$$\begin{tikzcd} E_0 \arrow[r, "\bar{f}"] \arrow[d, "p_0"] & E_1 \arrow[d, "p_1"] \\ X_0 \arrow[r, "f"] & X_1 \end{tikzcd} \tag{2.6}$$

The composition of fiber maps $(f, \bar{f}), (g, \bar{g})$ on suitable spaces is $(f, \bar{f})(g, \bar{g}) = (fg, \bar{f}\bar{g})$

A *fiber homotopy* $(h, \bar{h})$ is a pair of maps $h : X_0 \times I \to X_1$ and $\bar{h} : E_0 \times I \to E_1$ such that for any $t \in I$ is $(h_t, \bar{h}_t)$ a fiber map.

A *strong fiber homotopy* is a fiber homotopy $(h, \bar{h})$ such that $h_t : X_0 \to X_1$ is stationary for $t \in I$.

If $X_0 = X_1$ the fiber map $(\mathrm{id}_{X_0}, \bar{f}) : p_0 \to p_1$ is a *strong fiber homotopy equivalence* if there is a fiber map $(\mathrm{id}_{X_0}, \bar{g}) : p_1 \to p_0$ such that both compositions are strong fiber homotopic to identity. Denote this equivalence relation by $p_0 \simeq_{\mathrm{sfh}} p_1$.

The strong fiber homotopy equivalence turns out to be the most suitable candidate for what it means that two spherical fibrations are the "same".*

The definition of strong fiber homotopy equivalence is not very practical in proofs, so this is a sufficient (and also necessary) condition for a fiber map $(\mathrm{id}_X, \overline{f})$ of a fibration to be a strong fiber homotopy equivalence.

Lemma 2.1.11. *[1, Lemma 5.98, p. 177] Let $p_i : E_i \to X$, $i = 0, 1$ be two fibrations over the same CW complex X. Let $\overline{f} : E_0 \to E_1$ be a map such that $p_0 = p_1 \circ \overline{f}$ (in other words $(\mathrm{id}_X, \overline{f})$ is a fiber map). Suppose that for every $x \in X$ the map $\overline{f}$ restricts to a homotopy equivalence $p_0^{-1}(x) \to p_1^{-1}(x)$. Then $(\mathrm{id}_X, \overline{f})$ is a strong fiber homotopy equivalence.*

A result similar to the Lemma 2.1.11 holds for vector bundles. Namely let $p_1 : E_1 \to X$, $p_2 : E_2 \to X$ be two vector bundles of the same dimension over the same base space. A continuous map $h : E_1 \to E_2$ taking each fiber $p_1^{-1}(x)$ by a linear isomorphism to the fiber $p_2^{-1}(x)$ for each $x \in X$ is a vector bundle isomorphism (see, e.g. [6, Lemma 1.1, p. 8]).

Definition 2.1.12. [1, p. 138] Let $p : E \to X$ be a $(k-1)$-spherical fibration over some finite CW complex X. Let $[p]$ denote the equivalence class of fibrations strong fiber homotopy equivalent to p. Define

$$\mathrm{SF}_k(X) = \{[p] |\, p : E \to X \text{ is a } (k-1)\text{-spherical fibration}\} \tag{2.7}$$

Definition 2.1.13. Let X be a finite CW complex. Let the set of isomorphism classes of real k-dimensional vector bundles over X be denoted by

$$\mathrm{VB}_k(X) = \{[p] |\, p : E \to X \text{ is a } k\text{-dimensional real vector bundle}\} \tag{2.8}$$

We will need some triviality results for fibrations over the contractible space.

Proposition 2.1.14. *[5, Proposition 4.62] Let $p : E \to X$ be a spherical fibration, Y be a space and $f_0 : Y \to X$, $f_1 : Y \to X$ be two homotopic maps. Then the fibrations $f_0^*(E) \to Y$ and $f_1^*(E) \to Y$ are strong fiber homotopy equivalent.*

Corollary 2.1.15. *Let $p : E \to X$ be a $(k-1)$-spherical fibration for X a contractible space. Then p is strong fiber homotopy equivalent to a trivial fibration $S^{k-1} \times X \to X$*

*A bit on the notation. What Hatcher calls a fiber map we call a fiber map $(id_B, \bar{f})$ (of spherical fibrations) over the same base spaces. What Hatcher calls a fiber homotopy equivalence we call (for a spherical fibration) a strong fiber homotopy equivalence.

Proof. Since X is contractible, we have that the constant map $* : X \to X$ and the identity id_X are homotopic. From the Proposition (2.1.14), we have that the pullbacks are strong fiber homotopy equivalent. Pullback along a constant map is a trivial fibration, while the pullback along the identity is the original fibration p. □

Note that the previous Proposition holds for non-spherical fibrations too with similarly defined strong fiber homotopy equivalence (see, e.g. [5, p. 406 and Corollary 4.63]). An analogous claim holds for vector bundles and vector bundle isomorphisms.

Note 2.1.16. Let $F \to E \to X$ be any fibration. Later in this thesis, we will be assuming that fibers of our fibrations are simply connected or even spheres S^k (for $k > 1$). We will also be assuming that either base spaces or total spaces are simply connected. From these assumptions, one can infer information about homotopy groups of the remaining spaces in the fibration by the means of the long exact sequence of homotopy groups (or sets for π_0)

$$\cdots \longrightarrow \pi_1(F) \longrightarrow \pi_1(E) \longrightarrow \pi_1(X) \longrightarrow \pi_0(F) \longrightarrow \pi_0(E) \tag{2.9}$$

From simple connectedness of F we get $\pi_1(E) \cong \pi_1(X)$. In other words, the simple connectedness of either E or X implies the same for the other.

If X is simply connected, E connected then F is connected.

There is a notion of Cartesian product for vector bundles (over possibly different spaces) and Whitney sum for vector bundles over the same base space.

Namely given two vector bundles $p_1 : E_1 \to X_1$, $p_2 : E_2 \to X_2$ define a *Cartesian product* as a map $p_1 \times p_2 : E_1 \times E_2 \to X_1 \times X_2$ such $(e_1, e_2) \mapsto (p_1(e_1), p_2(e_2))$. This is a vector bundle with fiber $p_1^{-1}(x_1) \times p_2^{-1}(x_2)$.

If $X = X_1 = X_2$ there is the diagonal map $\Delta : X \to X \times X$, $x \mapsto (x, x)$. Define the *Whitney sum* $p_1 \oplus p_2$ to be the pullback of $p_1 \times p_2$ along the diagonal map Δ. The Whitney sum of two vector bundles over X is then again a vector bundle over X.

For all these proofs see [11, p. 27].

Example 2.1.17. Our main example of a Whitney sum will be the Whitney sum of a vector bundle with a trivial vector bundle. This special case is referred to as the stabilisations of vector bundles. This is the topic of the section 2.2.

Let M be an n-dimensional compact manifold embedded in a Euclidean space $i_{n+k} : M \to \mathbb{R}^{n+k}$. We have the tubular neighbourhood, part of which is the normal bundle $\nu(i_{n+k})$ over M. We may chose to embed M in a larger space $i_{n+k+1} : M \to$

$\mathbb{R}^{n+k+1}$ by a postcomposition with the natural inclusion $\mathbb{R}^{n+k} \to \mathbb{R}^{n+k+1}$. Then, for the normal bundle of the embedding i_{n+k+1} we have $\nu(i_{n+k}) \oplus \underline{\mathbb{R}} \cong \nu(i_{n+k+1})$ for $\underline{\mathbb{R}}$ a trivial one dimensional vector bundle over M.

Now we will define analogous constructions for spherical fibrations. The same construction would not work because the product, as defined above for vector bundles, would not yield a spherical fibration. Rather the fiber of $p_1 \times p_2$ and $p_1 \oplus p_2$ would be $S^{k-1} \times S^{l-1}$.

The *join* of two spaces X and Y is the space $X * Y = X \times Y \times I/\sim$ for an equivalence relation defined by $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$ for all $x_1, x_2, x \in X$ and $y_1, y_2, y \in Y$.

It can be defined equivalently as the following pushout:

$$\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] & \mathrm{Cone}(X) \times Y \arrow[d] \\
X \times \mathrm{Cone}(Y) \arrow[r] & X * Y
\end{tikzcd} \tag{2.10}$$

The following will be the reason why this is a suitable construction for combining two spherical fibrations together.

Lemma 2.1.18. *For any $k, l \geq 0$ we have $S^k * S^l \cong S^{k+l+1}$.*

Proof. We are looking for the pushout for the following diagram

$$\begin{tikzcd}
S^k \times S^l \arrow[r] \arrow[d] & D^{k+1} \times S^l \\
S^k \times D^{l+1} &
\end{tikzcd} \tag{2.11}$$

Such a pushout is a space $D^{k+1} \times S^l \cup_{S^k \times S^l} S^k \times D^{l+1}$ which is just the boundary of D^{k+l+2} when the disk is written as $D^{k+1} \times D^{l+1}$. So we have $S^k * S^l \cong \partial D^{k+l+2} \cong S^{k+l+1}$. □

Definition 2.1.19. [1, p. 137] Assume $k, l \geq 1$ and let $p_1 : E_1 \to X$ be a $(k-1)$-spherical fibration and $p_2 : E_2 \to Y$ be an $(l-1)$- spherical fibration .

The *exterior fiberwise join* is a $(k+l-1)$- spherical fibration

$$p_1 *_e p_2 : E_1 *_e E_2 \to X \times Y \tag{2.12}$$

Where $E_1 *_e E_2 = E_1 \times E_2 \times I/\sim$ for en equivalence relation $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$ for all $x_1, x_2, x \in X$ such that $p_1(x_1) = p_1(x_2)$ and $y_1, y_2, y \in Y$

such that $p_2(y_1) = p_2(y_2)$. These last two conditions justify the additional adjective *fiberwise*.

The projection 2.12 is defined on $E_1 \times E_2 \times I$ by $(e_1, e_2, t) \mapsto (p_1(x), p_2(y))$ for any $x \in X$, $y \in Y$, $t \in T$. The elements which are equivalent under the equivalence relation $\sim$ get mapped to the same element, so the map $p_1 *_e p_2$ is well defined on $E_1 *_e E_2$.

For the proof that 2.12 is a fibration, see the cited source. To show that the fibration 2.12 has a spherical fiber take a preimage

$$\begin{aligned}(p_1 *_e p_2)^{-1}(x, y) &= \{[(e_1, e_2, t)] |\, p_1(e_1) = x, p_2(e_2) = y, t \in I\}/\sim \\ &= p_1^{-1}(x) * p_2^{-1}(y) \simeq S^{k-1} * S^{l-1} \simeq S^{k+l-1}\end{aligned} \tag{2.13}$$

If $X = Y$ we have the *fiberwise join* a $(k + l - 1)$- spherical fibration $p_1 * p_2 = \Delta^*(p_1 *_e p_2)$, where $\Delta : X \to X \times X$ is the diagonal map $x \mapsto (x, x)$. Since Δ is injective, the total space of the fiberwise join of two fibrations $p_1 : E_1 \to X$, $p_2 : E_2 \to X$ is the preimage $(p_1 *_e p_2)^{-1}(\{(x, x) \in X \times X\})$. Hence the fiberwise join is the fibration

$$\begin{aligned}p_1 * p_2 : \{(e_1, e_2, t) |\, p_1(e_1) = p_2(e_2), t \in I\}/\sim &\to X \\ (e_1, e_2, t) &\mapsto p_1(e_1)\end{aligned} \tag{2.14}$$

Both of the fiberwise joins are associative up to strong fiber homotopy equivalence. The fiberwise join is natural with respect to pullbacks $f^*(p_1 * p_2) = f^*(p_1) * f^*(p_2)$, for some map $f : Y \to X$.

Now we will see how every k-dimensional vector bundle over a finite CW complex X can be thought of as a $(k - 1)$-spherical fibration. For a vector bundle $p : E \to X$ an inner product is a map $\langle -, - \rangle : E \oplus E \to \mathbb{R}$, which is a positive definite symmetric bilinear form on every fiber of p. Every vector bundle over a paracompact space has an inner product [6, Proposition 1.2, p. 11].

Definition 2.1.20. (Associated bundles/fibrations) From a k-dimensional vector bundle with an inner product over a finite CW complex $p : E \to X$, we can form the following bundles.

Define $DE = \{e \in E |\, \langle e, e \rangle \leq 1\}$ and $SE = \{e \in E |\, \langle e, e \rangle = 1\}$.

Then the restriction $Dp = p \restriction DE : DE \to X$ is a D^k-bundle. and $Sp = p \restriction SE : SE \to X$ is a S^{k-1}-bundle and so a $(k - 1)$-spherical fibration.

The latter construction defines a map $S : \mathrm{VB}_k(X) \to \mathrm{SF}_k(X)$.

We have the following compatibility condition for the map S when taking a Whitney sum with a trivial vector bundle and taking a fiberwise join with a trivial spherical fibration.

Lemma 2.1.21. *For any k-dimensional vector bundle $p : E \to X$ over a finite CW complex for any $l \geq 1$ we have*

$$S(E \oplus \underline{\mathbb{R}^l}) \simeq_{\mathrm{sfh}} S(E) * \underline{S^{l-1}}$$

Proof. It is enough to show it for $l = 1$, since then by induction, we have for $l > 1$

$$S(E \oplus \underline{\mathbb{R}^{l-1}} \oplus \underline{\mathbb{R}}) \simeq_{\mathrm{sfh}} S(E \oplus \underline{\mathbb{R}^{l-1}}) * \underline{S^0} \simeq_{\mathrm{sfh}} S(E) * S^{l-2} * S^0 \simeq_{\mathrm{sfh}} S(E) * S^{l-1}$$

where the last property is an application of the Lemma 2.1.18.

Take the inner product $\langle -, - \rangle : E \oplus E \to \mathbb{R}$ on the vector bundle p. Then we have an inner product $\langle -, - \rangle' : (E \oplus \underline{\mathbb{R}}) \oplus (E \oplus \underline{\mathbb{R}}) \to \mathbb{R}$ on the vector bundle $E \oplus \underline{\mathbb{R}}$ defined on each fiber by $\langle e + r, e' + r' \rangle' = \langle e, e' \rangle + rr'$.

Let us write the total spaces in question explicitly.

$$\begin{aligned} S(E) * \underline{S^0} = \{(e, x, s, t) |\ & s = \pm 1 \in S^0, x \in X, \\ & e \in p^{-1}(x) \subset E, \langle e, e \rangle = 1, t \in [0, 1]\} / \sim \end{aligned} \tag{2.15}$$

And the fibration

$$\begin{aligned} S(E) * \underline{S^0} &\to X \\ [(e, x, s, t)] &\mapsto p(e) \end{aligned} \tag{2.16}$$

The total space

$$S(E \oplus \underline{\mathbb{R}}) = \{(e, r) |\ e \in E, r \in \mathbb{R}, \langle e, e \rangle + r^2 = 1\} \tag{2.17}$$

of the fibration

$$
\begin{aligned}
S(E \oplus \underline{\mathbb{R}}) &\to X \\
(e, r) &\mapsto p(e)
\end{aligned}
$$

Define a map

$$
\begin{aligned}
\varphi : S(E) * \underline{S^0} &\to S(E \oplus \underline{\mathbb{R}}) \\
[(e, x, s, t)] &\mapsto ((\sqrt{1-t})e, \sqrt{t}s)
\end{aligned}
$$

This is obviously a fiber map because $p(e) = p((\sqrt{1-t})e)$. It is well defined (with respect to the equivalence relation $\sim$) since if $t = 0$ we have $(e, x, s_1, 0) \sim (e, x, s_1, 0)$ for $s_1, s_2 \in S^0$ the map φ agrees on those points $\varphi(e, x, s_1, 0) = \varphi(e, x, s_1, 0) = ((\sqrt{1-t})e, 0)$. On the other hand if $t = 1$ we have $(e_1, x, s, 1) \sim (e_2, x, s, 1)$ for $e_1, e_2 \in p^{-1}(x)$, but then $\varphi(e_1, x, s, 1) = \varphi(e_2, x, s, 1) = (0, s)$. The image of φ is in $S(E \oplus \underline{\mathbb{R}})$ since we have $\langle(\sqrt{1-t})e + \sqrt{t}s, (\sqrt{1-t})e, \sqrt{t}s\rangle' = \langle(\sqrt{1-t})e, (\sqrt{1-t})e\rangle + (\sqrt{t}s)^{=}(1-t) + t = 1$.

To show that φ is a strong fiber homotopy equivalence, according to the Lemma 2.1.11 we show that φ is a homotopy equivalence on each fiber. Take $x_0 \in X$. The fiber of the vector bundle is a k-dimensional vector space, so φ restricted to a single fiber gives a map $S(\mathbb{R}^k) * S^0 \to S(\mathbb{R}^k \oplus \mathbb{R})$ which is a map from a k-sphere to a k-sphere defined by $[(e, x_0, s, t)] \mapsto ((\sqrt{1-t})e, \sqrt{t}s)$. We claim that φ is a bijective, continuous map. For if we take an element $(e', r) \in S(\mathbb{R}^k \oplus \mathbb{R})$ it is an image under φ of a single element namely $(\frac{e'}{\sqrt{1-t}}, x_0, \frac{r}{\sqrt{t}}, t)$ for $t = |r| \in \langle 0, 1\rangle$. Hence it is a homotopy equivalence on fibers. $\square$

Definition 2.1.22. [1, p. 136] Let $p : E \to X$ be a $(k-1)$-spherical fibration. Then the *associated disk fibration* is a fibration $Dp : DE = \mathrm{Cyl}(p) \to X$, where $\mathrm{Cyl}(p)$ is the mapping cylinder.

Definition 2.1.23. [1, p. 136] Given a vector bundle $q : E \to X$ one may define the Thom space $\mathrm{Th}(q)$ to be the pointed space DE/SE, with SE/SE the basepoint.

Similarly, given a spherical fibration $p : E \to X$ the Thom space is the pointed space $\mathrm{Th}(p) = DE/E$. This is homeomorphic to $\mathrm{Cone}(p)$, the mapping cone of p.

2.2 The Stable Vector Bundle and the Stable Spherical Fibration

Recall that we are assuming that the bases of bundles/fibrations are finite CW complexes. This implies being Hausdorff and compact, which is the condition required in some of the proofs we cite.

Let us take two integers $0 < k \leq l$. We have a natural map $\mathrm{VB}_k(X) \to \mathrm{VB}_l(X)$ defined by stabilisation $[p] \to [\underline{R^{l-k}} \oplus p]$, taking the Whitney sum with a trivial $(l-k)$-dimensional vector bundle $\underline{R^{l-k}}$.

Now we define the set of *stable isomorphism classes* of vector bundles over X as

$$\mathrm{VB}(X) = \mathrm{colim}_{k \to \infty} \mathrm{VB}_k(X) \tag{2.18}$$

A *stable vector bundle* over X is an element of $\mathrm{VB}(X)$.

From the definition of a colimit it follows that there are maps $\mathrm{VB}_k(X) \to \mathrm{VB}(X)$. So any vector bundle $[p] \in \mathrm{VB}_k(X)$ can be thought of as a class $[p] \in \mathrm{VB}(X)$. Two vector bundles $[p_{k_1}] \in \mathrm{VB}_{k_1}(X)$ and $[p_{k_2}] \in \mathrm{VB}_{k_2}(X)$ represent the same class $[p_{k_1}] = [p_{k_2}]$ in $\mathrm{VB}(X)$ if and only if there exists $M \geq k_1, k_2$ such that $[p_{k_1} \oplus \underline{\mathbb{R}^{M-k_1}}] \cong [p_{k_2} \oplus \underline{\mathbb{R}^{M-k_2}}]$.

An analogous definition can be made for spherical fibration and the operation of fiberwise join.

For $0 < k < l$ we have maps $\mathrm{SF}_k(X) \to \mathrm{SF}_l(X)$ defined by stabilisation $[p] \mapsto [\underline{S^{l-k-1}} * p]$. The image of this map is a class of $(l-1)$-spherical fibrations because we had for dimensions that $S^{k-1} * S^{l-k-1} = S^{((k-1)+(l-k-1)+1)} = S^{l-1}$ (recall that $\mathrm{SF}_l(X)$ was defined to be the equivalence classes of $(l-1)$-spherical fibrations).

Now define

$$\mathrm{SF}(X) = \mathrm{colim}_{k \to \infty} \mathrm{SF}_k(X) \tag{2.19}$$

Just like with vector bundles, from the colimit we have the maps $\mathrm{SF}_k(X) \to \mathrm{SF}(X)$ and thus any spherical fibration $[p] \in SF_k(X)$ can be thought of as representing a class in $\mathrm{SF}(X)$. Two spherical fibrations $[p_{k_1}] \in \mathrm{SF}_{k_1}(X)$ and $[p_{k_2}] \in \mathrm{SF}_{k_2}(X)$ represent the same class $[p_{k_1}] = [p_{k_2}]$ in $\mathrm{SF}(X)$ if and only if there exists $M \geq k_1, k_2$ such that $\underline{S^{M-k_1}} * p_{k_1} \simeq_{\mathrm{sfh}} \underline{S^{M-k_2}} * p_{k_2}$. Such two fibrations are then called *stably fiber homotopy equivalent*.

There is, it turns out, a structure of an abelian group on the sets $\mathrm{VB}(X)$, $\mathrm{SF}(X)$.

Lemma 2.2.1. *[6, Proposition 2.9, p. 39] The set* $\mathrm{VB}(X)$ *with the operation* $\oplus$ *has the structure of an abelian group.*

For example, the identity is the class $[\underline{\mathbb{R}^k}]$ represented by any trivial vector bundle over X.

Note 2.2.2. In the Section 1.3 the tubular neighbourhood was defined for an embedding i of a manifold into a Euclidean space and this yielded $\nu(i)$ a normal bundle of M. The tangent bundle of a manifold does not depend on any choice. But the Whitney sum of $TM \oplus \nu(i)$ is a trivial vector bundle. So the stabilisation of a normal bundle is an inverse to the stabilisation of the tangent bundle. Inverses are unique in groups. That shows that the stabilisation of $\nu(i)$ is independent of the choice of i.

Lemma 2.2.3. *The set* $\mathrm{SF}(X)$ *with the operation* $*$ *has the structure of a finite abelian group.*

For the proof that $\mathrm{SF}(X)$ is an abelian group see [1, Section 5.2, in particular, Lemma 5.23] and to see that it is a finite group see [1, Corollary 5.36, p. 147].

The identity element of $\mathrm{SF}(X)$ is likewise the class of trivial spherical fibrations $[\underline{S^0}]$ over X.

In the definition 2.1.20 we constructed an associated $(k-1)$-spherical fibration to a k-dimensional vector bundle. Now we can show that the same construction is well defined on stable fibrations/stable vector bundles.

Proposition 2.2.4. *There is a map from stable vector bundles over* X *and stable spherical fibrations over* X *given by constructing the associated spherical fibration of the definition 2.1.20.*

$$\begin{aligned} S : \mathrm{VB}(X) &\to \mathrm{SF}(X) \\ [p] &\mapsto [Sp] \end{aligned} \tag{2.20}$$

Proof. All we need to do is to verify that it is well defined under stabilisation. For this, we use the Lemma 2.1.21 which gives compatibility of stabilisations and the associated spherical fibrations. For a stable vector bundle $[p] \in \mathrm{VB}(X)$ pick another representative $q \in [p]$, $q \oplus \underline{\mathbb{R}^M} \cong p \oplus \underline{\mathbb{R}^N}$. Now in the group $\mathrm{SF}(X)$ we have

$$Sp * \underline{S^{N-1}} \simeq_{\mathrm{sfh}} S(p \oplus \underline{\mathbb{R}^N}) \simeq_{\mathrm{sfh}} S(q \oplus \underline{\mathbb{R}^M}) \simeq_{\mathrm{sfh}} Sq * \underline{S^{M-1}}$$

Thus $[Sp] = [Sq]$ in $\mathrm{SF}(X)$. ∎

2.3 The Thom Isomorphism Theorem

Consider a k-dimensional vector bundle or a $(k-1)$-spherical fibration for $k > 0$. Just like with Poincaré duality we will define a notion of orientability of the vector bundle/ spherical fibration and discuss how in our simply connected case we will get orientability trivially. Then there will exist a *Thom class* $U_p \in H^k(\mathrm{Th}(p))$ where $\mathrm{Th}(p)$ is the Thom space of a vector bundle or of a spherical fibration defined in 2.1.23. For this Thom class we will then get the Thom isomorphism Theorem 2.3.5.

From now we will work with spherical fibrations for simplicity of notation. To get the result for vector bundles, one forms the associated spherical bundle as defined in 2.1.20.

Now we will define an orientability for a spherical fibration (following [1, p. 138]). Let $p : E \to X$ be a $(k-1)$-spherical fibration and γ a path in X between the points x and y from X $\gamma : I \to X$. A *fiber transport* along γ is a way to define a map $p^{-1}(x) \to p^{-1}(y)$. Take an inclusion $p^{-1}(x) \to E$ and a map $p^{-1}(x) \times I \to X$, $(z,t) \mapsto \gamma(t)$. Now we have a diagram

$$\begin{tikzcd} p^{-1}(x) \arrow[r] \arrow[d, "z\mapsto(z{,}0)"'] & E \arrow[d, "p"] \\ p^{-1}(x) \times I \arrow[r] \arrow[ur, dashed, "t"] & X \end{tikzcd} \tag{2.21}$$

and since p is a fibration, by the homotopy lifting property we have a map $t : p^{-1}(x) \times I \to E$. By commutativity of the diagram it is true that $t_1 : p^{-1}(x) \to p^{-1}(y)$ which is the desired map. If for γ we only consider loops at some $x \in X$ we get the following morphism $t_x : \pi_1(X,x) \to [p^{-1}(x), p^{-1}(x)]$.

Definition 2.3.1. A $(k-1)$-spherical fibration is called *orientable* if the map t_x is trivial for every $x \in X$. In other words, it is orientable if every fiber transport along every loop in any point yields a map of fibers homotopic to the identity.

Thus we get our simplification:

Corollary 2.3.2. *Any spherical fibration over simply connected space is orientable.*

Let a spherical fibration $p : E \to X$ be orientable and let $x \in X$ be any point. We can choose and fix an identification

$$H^k(DE_x, E_x) \xrightarrow{\cong} H^k(D^k, S^{k-1}) \tag{2.22}$$

where $E_x = p^{-1}(x)$ and $DE_x = (Dp)^{-1}(x)$ for the associated disk fibration.

Note that the choice of 2.22 is equivalent to the choice of the generator of $H^k(DE_x, E_x)$ as the preimage of the canonical generator of $H^k(D^k, S^{k-1})$.

For a different choice $y \in X$ we have the the canonical identification $H^k(DE_y, E_y) \xrightarrow{\cong} H^k(D^k, S^{k-1})$ defined by precomposing 2.22 with the map induced by fiber transport map t_1 along some path from x to y. The choice of this path does not matter since $t_x : \pi_1(X, x) \to [p^{-1}(x), p^{-1}(x)]$ is trivial.

We will not prove the following Proposition. The reader can find the proof in the cited reference.

Proposition 2.3.3. *[1, Lemma 5.42, p. 149] Let $p : E \to X$ be an orientable $(k-1)$-spherical fibration. Take any $x \in X$. Then the following map is an isomorphism*

$$\alpha : H^k(DE, E) \xrightarrow{i^*} H^k(DE_x, E_x) \xrightarrow{\cong} H^k(D^k, S^{k-1}) \tag{2.23}$$

where the last arrow is the fixed identification 2.22.

Definition 2.3.4. Let $p : E \to X$ be an oriented $(k-1)$-spherical fibration and x a point in X. Form a *Thom class* $U_p \in H^k(DE, E)$ as a preimage of a chosen generator of $H^k(D^k, S^k)$ under the map α of the previous Proposition.

$$\alpha : H^k(DE, E) \xrightarrow{i^*} H^k(DE_x, E_x) \xrightarrow{\cong} H^k(D^k, S^{k-1}) \tag{2.24}$$

Theorem 2.3.5. *(Thom isomorphism Theorem)[1, Theorem 5.52] Let $p : E \to X$ be a $(k-1)$-spherical fibration over a simply connected finite CW complex X. Then the following compositions are isomorphisms of groups for any n.*

$$H_{n+k}(DE, E) \xrightarrow{U_p \frown -} H_n(DE) \xrightarrow{Dp^*} H_n(X) \tag{2.25}$$

$$H^n(X) \xrightarrow{Dp_*} H^n(DE) \xrightarrow{- \smile U_p} H^{n+k}(DE, E) \tag{2.26}$$

Proof. In the cited reference the authors leave the proof for the case of the trivial fibrations to the reader. Assuming the trivial case is true, the proof goes by the induction on the number of cells in X. We will show the proof for the trivial fibration for both the homological version and the cohomological version.

Let us start with the homological version 2.25.

Let p be a trivial fibration $p : S^{k-1} \times X \to X$. Now we have $(DE, E) = (D^k \times X, S^{k-1} \times X)$. Pick any point $x \in X$.

The Thom class U_p for a trivial fibration is the preimage of the generator in the isomorphism.

$$i^* : H^k(D^k \times X, S^{k-1} \times X) \to H^k(D^k, S^{k-1})$$

Where $i : (D^k, S^{k-1}) \to (D^k \times \{x\}, S^{k-1} \times \{x\}) \to (D^k \times X, S^{k-1} \times X)$ is an inclusion.

Since the pair (D^k, S^{k-1}) has a nonzero (co)homology only in the dimension k, by the Künneth formula we have an isomorphism

$$H^k(D^k \times X, S^{k-1} \times X) \xleftarrow{\cong} H^k(D^k, S^{k-1}) \otimes H^0(X)$$
$$a \times b \leftarrow\!\shortmid (a, b)$$

For the unit 1_X of the cohomology ring $H^*(X)$ we know that the mapping $a \mapsto a \times 1_X$ is an isomorphism. It is also an inverse to i^* (this follows from the definition of the Künneth homomorphism by the cup products. Namely $a \times 1_X$ can be written as $p_1^*(a) \smile p_2^*(1_X) = p_1^*(a)$ for the projections $p_1 : D^k \times X \to D^k$ and $p_2 : D^k \times X \to X$. We have $i^* p_1^* = (p_1 i)^* = \mathrm{id}^* = \mathrm{id}$. So also $(p_1 i)_* = \mathrm{id}$). The group $H^k(D^k, S^{k-1})$ is infinite cyclic with the canonical generator e' and hence the Thom class is $U_p = e' \times 1_X \in H^k(D^k, S^{k-1}) \otimes H_0(X)$.

We need to prove that the map $H_{n+k}(D^k \times X, S^{k-1} \times X) \xrightarrow{U_p \frown -} H_n(D^k \times X) \xrightarrow{Dp^*} H_n(X)$ is an isomorphism. The second map is induced by the homotopy equivalence $Dp : X \to D^k \times X$ and hence is an isomorphism.

The group $H_{n+k}(D^k \times X, S^{k-1} \times X)$ can also be decomposed by the Künneth formula

$$H_{n+k}(D^k \times X, S^{k-1} \times X) \xleftarrow{\cong} H_k(D^k, S^{k-1}) \otimes H_n(X) \cong H_n(X)$$
$$a \times b \leftarrow\!\shortmid (a, b)$$

To any element $z \in H_{n+k}(D^k \times X, S^{k-1} \times X)$ there exists a unique element $z_x \in H_n(X)$ such that z can be written as $z = e \times z_X \in H_k(D^k, S^{k-1}) \otimes H_n(X)$, for e a generator of $H_k(D^k, S^{k-1})$. The elements z and z_x are in a one-to-one correspondence.

Now we will use a "distributive" property of the cross product and the cup product (see for instance [9, p. 126]):

$$U_p \frown z = (e' \times 1_X) \frown (e \times z_X) = (-1)^{|e||1_X|}(e' \frown e) \times (1_X \frown z_X) = \pm\, (e' \frown e) \times z_X \quad (2.27)$$

The element $(e' \frown e)$ belongs to the group $H_0(D^k)$. From properties of the cap product we have in this case $\epsilon(e' \frown e) = e'(e) = 1$.†

It follows that $U_p \frown z = \pm z_x$. The elements z and $\pm z$ are (for either sign) in a one-to-one correspondence and so the Thom homomorphism 2.25 is bijective.

In the cohomological version 2.26 we have the same Thom class $U_p = e' \times 1_X$. The typical element of $H^n(D^k \times X)$ is $1_* \times z_x$ for $1_* \in H^0(D^k)$ the unit in the cohomological ring of D^k and $z_x \in H^n(X)$. Now by another "distributive property" [9, p. 126] we have

$$(1_* \times z_X) \smile (e' \times 1_X) = (-1)^{|z_x||e'|}(1_* \smile e') \times (z_x \smile 1_x) = \pm e' \times z_x \quad (2.28)$$

The target group is $H^{n+k}(D^k \times X, S^{k-1} \times X)$ and its typical element (by the Künneth formula) is what we have just obtained $e' \times z_x$. Thus the cohomological Thom homomorphism is bijective and hence an isomorphism. □

The following Theorem can be viewed as a converse to the Thom isomorphism Theorem. Its proof is rather complicated and can be found in the cited reference.

Theorem 2.3.6. *[1, Theorem 5.60, p. 155.] Let $p : E \to X$ be a fibration with both E, X and the fiber simply connected. Let $k \geq 3$. Suppose there exists an element $u \in H^{n+k}(DE, E)$ such that the map*

$$H_{n+k}(DE, E) \xrightarrow{u \frown -} H_n(DE) \xrightarrow{Dp^*} H_n(X)$$

is an isomorphism for all n. Then the fiber of p is homotopy equivalent to S^{k-1}.

By the Note 2.1.16 if the fiber is simply connected and the spaces E, X are connected, we only need to assume one of the spaces E or X be simply connected for both of them to be.

†Here ϵ is the augmentation homomorphism $\epsilon : H_0(X) \to \mathbb{Z}$ defined as $\sum n_i x_i \mapsto \sum n_i$ for 0-simplices $x \in C_0$. It can be shown that if X is connected ϵ is an isomorphism (see e.g. [3, p. 172]).

Chapter 3

Spivak Normal Structure

Definition 3.0.1. (Spivak normal $(k-1)$-structure)[1, Definition 5.65, p. 161] Let X be an n-dimensional Poincaré complex. The *Spivak normal* $(k-1)$*-structure* is a pair (p, c) where $p : E \to X$ is a $(k-1)$-spherical fibration called *Spivak normal fibration* and c is a map $c : S^{n+k} \to \mathrm{Th}(p)$ such that the Hurewicz homomorphism $h : \pi_{n+k}(\mathrm{Th}(p)) \to H_{n+k}(\mathrm{Th}(p))$ sends the class $[p]$ to a generator of $H_{n+k}(\mathrm{Th}(p)) \cong H_{n+k}(DE, E)$.

One should look at the Spivak normal fibration of a finite Poincaré complex as an analogy of the normal vector bundle for a manifold. In this chapter, we will follow [1] and [12] in showing the existence of *Spivak normal structure* for simply connected finite CW complexes. There is a uniqueness result that will be stated and not proven, and the proof can be found in, e.g. [1, Chapter 5.].

3.1 The Thom-Pontryagin Construction

In this section, we construct a Spivak normal structure for compact manifolds.

Let M be an n-dimensional compact manifold with an embedding $i : M \to \mathbb{R}^{n+k}$ into some Euclidean space. We have the tubular neighbourhood (f, ν) where $\nu : E \to M$ is a normal vector bundle of M and $f : E \to \mathbb{R}^{n+k}$ is an embedding of its total space.

Take the associated disk bundle $D\nu : DE \to M$ and the associated sphere bundle $S\nu : SE \to M$. The map f restricts to embeddings on DE and SE to give a regular neighbourhood, a manifold with boundary $(f(DE), f(SE))$ of M. Let us drop the f and identify (DE, SE) with its image in $\mathbb{R}^{n+k}$. The Thom space of $S\nu$ is by definition

$$\mathrm{Th}(S\nu) = DE/SE \tag{3.1}$$

Let $(\mathbb{R}^{n+k})^c$ is the one point compactification of the space $\mathbb{R}^{n+k}$. It is homeomorphic to S^{n+k}.

Define the Thom collapse map

$$c : (\mathbb{R}^{n+k})^c \to DE/SE = \mathrm{Th}(S\nu) \tag{3.2}$$

as an identity on the interior of DE and the constant map $(\mathbb{R}^{n+k})^c \setminus \mathrm{int}(DE) \mapsto SE/SE$ on the rest.

Lemma 3.1.1. *(Spivak normal structure for manifolds)[1, Example 5.66, p. 162] Let M be a compact n-dimensional manifold. In the situation above, $(S\nu, c)$ for $S\nu : SE \to X$ the spherical fibration associated to a normal vector bundle of M and the Thom collapse map $c : S^{n+k} \to \mathrm{Th}(S\nu)$ of 3.2 form a Spivak normal $(k-1)$-structure of M.*

Proof. The group $H_{n+k}(\mathrm{Th}(S\nu)) \cong H_{n+k}(DE, SE)$ is infinite cyclic because by the Thom isomorphism Theorem 2.3.5, it is isomorphic to $H_n(M)$. Since X is orientable and connected, from the Poincaré duality we have $H_n(M) \cong H^0(M)$. From the universal coefficient Theorem we have $H^0(M) \cong H_0(X) \cong \mathbb{Z}$.

We need to show that the image of the element $[c] \in \pi_{n+k}(\mathrm{Th}(S\nu))$ under the Hurewicz homomorphism is the generator of the group $H_{n+k}(\mathrm{Th}(S\nu))$. Let us recall the Hurewicz homomorphism

$$\begin{aligned} h : \pi_{n+k}(\mathrm{Th}(S\nu)) &\to H_{n+k}(\mathrm{Th}(S\nu)) \\ [c] &\mapsto c_*([S^{n+k}]) \end{aligned} \tag{3.3}$$

where $[S^{n+k}] \in H_{n+k}(S^{n+k})$ is the fundamental class of S^{n+k}.

We have identified $\mathrm{Th}(S\nu)$ with DE/SE in 3.1. In the homology (since $n+k > 0$) we even have $H_{n+k}(DE/SE) \cong H_{n+k}(DE, SE)$. The pair (DE, SE) is a manifold with a boundary and hence has a fundamental class $[DE, SE]$ which we now fix. Now we have $c_*([S^{n+k}]) = m[DE, SE]$ for m the degree of c. The degree* of a mapping is the sum of its local degrees at some regular value. Recall that c was the collapse map $(\mathbb{R}^{n+k})^c \to DE/SE$. We can take any point $y \in DE\setminus SE$. It is a regular value of c since the collapse map is an identity in the interior of DE. The point y has a single preimage under c. Thus $m = \pm 1$ and the Hurewicz homomorphism maps $[c]$ to a generator.

□

*The degree of mapping between two n-dimensional oriented manifolds $f : M \to N$ can be defined in two equivalent ways. It is either the sum of the local degrees of the mapping ($\deg(f) = \sum_{x \in f^{-1}(y)} \mathrm{sign}(df_x)$ for any regular value $y \in N$ of f) or as the number m in the equation $f_*([M]) = m[N]$ for fundamental classes $[M]$, $[N]$ of M and N respectively.

3.2 The Construction

Let X be a finite, n-dimensional CW complex. There exists an embedding of X into some Euclidean space $i_X : X \to \mathbb{R}^{n+k}$. Take a regular neighbourhood $(N, \partial N)$ of $i_X(X)$ (see definition 1.3.4). We will construct a particular fibration that will later turn out to be spherical assuming the space X is a Poincaré complex. We will be a little sloppy here and identify the spaces which are homotopy equivalent. This is done because it makes it easier to see what is going on. This construction is from [1, pp. 158-161] and there one finds all the details.

The following is a way of turning any map into a fibration, by a so-called pathspace construction.

Lemma 3.2.1. *(The pathspace construction)[5, p. 407] Let $f : A \to B$ be any map of spaces. Let E_f be the space of all pairs (a, γ) for a point $a \in A$ and a path $\gamma : I \to B$ such that $\gamma(0) = f(a)$. The space E_f is topologized as a subspace of $A \times B^I$ †. Let $p : E_f \to B$ be the map defined by $p(a, \gamma) = \gamma(1)$.*

Then p is a fibration. The map $E_f \xrightarrow{\simeq} A$ given by $(a, \gamma) \mapsto a$ is a homotopy equivalence and the map $A \xrightarrow{\simeq} E_f$ given by $a \mapsto (a, *)$ for $*$ a constant path at $f(a)$ is its homotopy inverse.

The fiber of p is referred to as the homotopy fiber of the map f.

A way to restate the previous Lemma is to say that every map $f : A \to B$ can be decomposed into a homotopy equivalence and a fibration $f : A \xrightarrow{\simeq} E_f \xrightarrow{p} B$.

Let $i_N : \partial N \hookrightarrow N$ be the inclusion and denote associated fibration by

$$\begin{aligned} q_N : E_N &\to N \\ (y, w) &\mapsto w(1) \end{aligned} \tag{3.4}$$

The total space is $E_N = \{(y, w) |\ y \in \partial N, w : I \to N, w(0) = i_N(y)\} \simeq \partial N$.

From the definition of a regular neighbourhood, the inclusion $i_X : X \to N$ is a homotopy equivalence. Define a fibration $p_N : S_N \to X$, as the pullback of q_N along i_X.

$$\begin{tikzcd} S_N = i_X^*(E_N) \arrow[r] \arrow[d, "p_N"] & E_N \arrow[d, "q_N"] \\ X \arrow[r, "i_X"] & N \end{tikzcd} \tag{3.5}$$

†The space B^I is the space of maps from I to B with the usual compact-open topology.

Since i_X is a homotopy equivalence, so is the map $S_N \to E_N$. So we have homotopy the equivalences $S_N \xrightarrow{\simeq} E_N \xrightarrow{\simeq} \partial N$. Now p_N is the fibration we wanted. To get the candidate for the Spivak normal structure, we also need the map $c : S^{n+k} \to \mathrm{Th}(p_N)$.

Consider the following diagram.

$$\begin{tikzcd} S_N \arrow[r, dash, "\simeq"] \arrow[d, "p_n"] & E_N \arrow[r, dash, "\simeq"] & \partial N \arrow[d, "i_N"] \\ X \arrow[rr, dash, "\simeq"] & & N \end{tikzcd} \tag{3.6}$$

It does not commute, but there is preferred homotopy between the two ways one can traverse from S_N to N, see [1, p. 160].

Thus we get a homotopy equivalence of the mapping cones of p_N and i_N.

$$\mathrm{Cone}(p_N) \xrightarrow{\simeq} \mathrm{Cone}(i_N) \tag{3.7}$$

The left space is the Thom space of p_N. The right space is homotopy equivalent to $N/\partial N$.

Thus we get a homotopy equivalence

$$v_N : \mathrm{Th}(p_N) \xrightarrow{\simeq} N/\partial N \tag{3.8}$$

Define the Thom collapse map as the composition

$$c : (\mathbb{R}^{n+k})^c \to N/\partial N \xrightarrow{\simeq} \mathrm{Th}(p_N) \tag{3.9}$$

where the first map is the same collapse map defined in 3.2.

This construction does not depend on the chosen regular neighbourhood in the following sense:

Assume we had another embedding $i'_X : X \to \mathbb{R}^{n+k}$ with a regular neighbourhood $(N', \partial N')$. From the uniqueness of regular neighbourhoods 1.3.5 it is possible to construct a map $\mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ which

- induces a strong fiber homotopy equivalence of the fibrations S_N,
- maps the elements c to each other by the induced map $\pi_{n+k}(\mathrm{Th}(p_N)) \xrightarrow{\cong} \pi_{n+k}(\mathrm{Th}(p_{N'}))$.

The proof of this claim can be found in [1, p. 160.].

3.3 The Spivak Normal Structure

Theorem 3.3.1. *[1, Theorem 5.87, p. 170] Let X be a finite simply connected Poincaré complex. Consider any embedding $i_X : X \hookrightarrow \mathbb{R}^{n+k}$ and its regular neighbourhood $(N, \partial N)$ for $k \geq 3$.*

Then X is a Poincaré complex if and only if the fiber of the normal fibration $p_N : S_N \to X$ constructed in the Section 3.2 is homotopy equivalent to S^k.

The goal of this section is to show the proof of the Theorem 3.3.1. This Theorem can simply be viewed as the assertion that for a simply connected Poincaré complex X the fibration $p_N : S_N \to X$ constructed in 3.2 is spherical. But the Theorem 3.3.1 actually shows an equivalence

$$X \text{ is a Poincaré complex} \Longleftrightarrow \text{The fibration } p_N \text{ is spherical}$$

.

Moreover, the Thom isomorphism Theorem 2.3.5 and its converse 2.3.6 add another equivalent condition

$$\begin{gathered} X \text{ is a Poincaré complex} \\ \Updownarrow \\ \text{The fibration } p_N \text{ is spherical} \\ \Updownarrow \\ \text{For fibration } p_N \text{ there is a Thom class for which the Thom isomorphism 2.3.5 holds} \end{gathered}$$

Proof of the theorem 3.3.1. First we show that the fiber F of $p_N : S_N \to X$ is simply connected. Let us first show that the pair $(N, \partial N)$ is 2-connected. Since $(N, \partial N)$ is a regular neighbourhood of X it follows that $(N, N \setminus X) \simeq (N, \partial N)$. Let us take any element of $\pi_i(N, N \setminus X)$ for $i \leq 2$ represented by a map of pairs $\varphi : (D^i, S^{i-1}) \to (N, N \setminus X)$. By universal position, since X is n-dimensional and $N \setminus X$ is at least $(n+3)$-dimensional, φ is homotopic to a map $\varphi : (D^i, S^{i-1}) \to (N \setminus X, N \setminus X)$.

Let us consider long exact sequences of homotopy groups of a pair $(N, \partial N)$ and of a fibration $F \to S_N \xrightarrow{p_N} X$. Here we use that $N \simeq X$, $S_N \simeq E_N \simeq \partial N$.

$$\cdots \longrightarrow \pi_2(\partial N) \xrightarrow{\iota} \pi_2(N) \longrightarrow \pi_1(F) \xrightarrow{0} \pi_1(\partial N) = 0 \tag{3.10}$$

$$\cdots \xrightarrow{\iota} \pi_2(\partial N) \longrightarrow \pi_2(N) \longrightarrow \pi_2(N, \partial N) \xrightarrow{0} \pi_1(\partial N) = 0 \tag{3.11}$$

We have $\pi_1(F) \cong \pi_2(N, \partial N) \cong 0$ because, in this case, those groups are completely determined by the rest of the sequence. For consider a long exact sequence, with y, X unknown.

$$\cdots \longrightarrow \pi_2(\partial N) \xrightarrow{\iota} \pi_2(N) \xrightarrow{y} X \xrightarrow{0} 0 \tag{3.12}$$

The map y is surjective with $\operatorname{Im} \iota = \operatorname{Ker} y = X$. Hence the map y and the group X are completely determined.

By the considerations in 2.1.16 F is connected. Thus the fiber F is simply connected.

From Poincaré duality, we obtain the following isomorphism.

$$- \frown [N, \partial N] : H^{n+k-*}(N, \partial N) \xrightarrow{\simeq} H_*(N) \tag{3.13}$$

for some fundamental class $[N, \partial N] \in H_{n+k}(N, \partial N)$.

Let us take any $u \in H^k(DS_N, S_N)$. We have a map used in the Thom isomorphism Theorem $H^{n-*}(X) \xrightarrow{\cong} H^{n-*}(DS_N) \xrightarrow{-\smile u} H^{n+k-*}(DS_N, S_N)$. By the Converse of the Thom isomorphism Theorem 2.3.6 we know that if there is such u that it is an isomorphism we have the fiber of $p_N : S_N \to X$ homotopy equivalent to the $(k-1)$-sphere.

For this element u let us define a candidate for the fundamental class of X, an element from $H_n(X)$. Take $u_X \in H_n(X)$ to be the image of $[N, \partial N]$ under the following map.

$$H_{n+k}(N, \partial N) \cong H_{n+k}(DS_N, S_N) \xrightarrow{u \frown -} H_n(DS_N) \cong H_n(X) \tag{3.14}$$

We shall write for the simplicity of notation that $u_X = u \frown [N, \partial N]$.

Then we claim that the following diagram commutes

$$\begin{tikzcd}
H^{n-*}(X) \arrow[r, "-\frown u_x"] \arrow[d, "-\smile u"] & H_*(X) \arrow[d, "\cong"] \\
H^{n+k-*}(N, \partial N) \arrow[r, "-\frown [N,\partial N]"] & H_*(N)
\end{tikzcd} \tag{3.15}$$

for the left vertical map being the composition

$$H^{n-*}(X) \cong H^{n-*}(DS_N) \xrightarrow{-\smile u} H^{n+k-*}(DS_N, S_N) \cong H^{n+k-*}(N, \partial N) \tag{3.16}$$

For upon fixing a value for $*$, take $\alpha \in H^{n-*}(X)$. Now we have to show that $(\alpha \smile u) \frown [N, \partial N]$ is equal to $\alpha \frown u_X = \alpha \frown (u \frown [N, \partial N])$. But this is true by a basic compatibility property of cup and cap products (see e.g. [3, Theorem 5.2, p. 336]).

Now assume that X is a Poincaré complex, with a fundamental class $[X]$. Then we have a composition (it is like going around the diagram 3.15 counterclockwise for $* = n$ and $u_X = [X]$)

$$(- \frown [X])^{-1} \circ (- \frown [N, \partial N]) : H^k(N, \partial N) \xrightarrow{\cong} H_n(N) \cong H_n(X) \xrightarrow{\cong} H^0(X) \quad (3.17)$$

The last group has a preferred generator 1 (a unit in the cohomology ring). Define $u \in H^k(N, \partial N)$ as the preimage of this unit. We check that if we defined u_x as in the equation 3.14 we would get $[X] = u_X$. From the same equation we have $u_X = u \frown [N, \partial N]$. From the equation 3.17 we have $u \frown [N, \partial N] = 1 \frown [X] = [X]$ and so $[X] = u_x$. Among other things we now know that the diagram 3.15 with the top horizontal map $(- \frown [X]) : H^{n-*}(X) \to H_*(X)$ commutes.

But then the map $- \smile u : H^{n-*}(X) \to H^{n+k-*}(N, \partial N)$ is an isomorphism. From the Converse of the Thom isomorphism Theorem 2.3.6, we have that the fiber of the fibration $p_N : S_N \to X$ is homotopy equivalent to a sphere.

On the other hand, assume that the fiber F of the fibration $p_N : S_N \to X$ is homotopy equivalent to S^{k-1}. By the Thom isomorphism Theorem 2.3.5, the Thom homomorphism is an isomorphism for the Thom class $u \in H^k(DS_N, S_N)$. From u construct $u_X \in H_n(X)$ as above from the map 3.14. But in the diagram 3.15 all the maps but the top horizontal are isomorphisms so that one is as well. Thus we have that X is a Poincaré complex with fundamental class u_X. □

Here is the statement of existence and uniqueness of the Spivak normal structure. The proof of the uniqueness can be found in [1, Section 5.].

Theorem 3.3.2. *[1, Theorem 5.68, p. 163] Let X be a finite n-dimensional Poincaré complex.*

i) Let k be an integer $k \geq n+1$. Then there exists a Spivak normal $(k-1)$-structure (p, c) defined in 3.0.1 (namely it is the structure (p_N, c) constructed in the Section 3.2).

ii) Let (p_i, c_i), $p_i : E_i \to X$ be two Spivak normal $(k_i - 1)$-structures, $c_i : S^{n+k_1} \to \mathrm{Th}(p_i)$ for $i = 0, 1$. Then there exists $k \geq k_0, k_1$ such that there is up to strong fiber homotopy a single strong fiber homotopy equivalence

$$(\mathrm{id}, \overline{f}) : p_0 * \underline{S^{k-k_0}} \to p_1 * \underline{S^{k-k_1}}$$

Such that $\pi_{n+k}(\mathrm{Th}(\overline{f}))(\sigma^{k-k_0}([c_0])) = \sigma^{k-k_1}([c_1])$.

We are, of course, working only with simply connected spaces, so we assume as much in the proof.

Proof of i) for the simply connected version. Let $k \geq n+1$. There is a an emebdding of X to a $(n+k+1)$-dimensional Euclidean space, see 1.3. Let us take (p_N, c) constructed in the Section 3.2. Since X is a Poincaré complex, Theorem 3.3.1 shows that the fibration p_N is a spherical fibration. From the Poincaré duality for X, and the universal coefficient theorem the top homology group $H_n(X)$ is infinite cyclic. By the Thom isomorphism Theorem 2.3.5 the group $H_{n+k}(\mathrm{Th}(p_N))$ is also infinite cyclic. We need to show that the image of the element $[c] \in \pi_{n+k}(\mathrm{Th}(p_N))$ under the Hurewicz homomorphism is the generator of the group $H_{n+k}(\mathrm{Th}(p_N))$. But we had a homotopy equivalence $\mathrm{Th}(p_N) \xrightarrow{\simeq} N/\partial N$ of 3.8. We identity the groups $\pi_{n+k}(\mathrm{Th}(p_N)) \cong \pi_{n+k}(N/\partial N)$ and $H_{n+k}(\mathrm{Th}(p_N))$ with $H_{n+k}(N, \partial N)$ (since $n+k > 0$). The proof then goes like in the case of a manifold, see 3.1.1.

□

Chapter 4

The Classification Theorems for Vector Bundles and Fibrations

4.1 The Clutching Construction for Vector Spaces

Vector bundles over spheres with k-dimensional fiber can be constructed by a *clutching construction*. For more details and proofs, see [6, p. 22ff]. The idea is that the base space S^n can be written as the gluing of the upper and the lower hemisphere $S^n = D^n_+ \cup_{S^{n-1}} D^n_-$. Now suppose we have a k-dimensional vector bundle over an n-sphere $p : E \to S^n$ then its restrictions (pullbacks) over the n-disks are necessarily trivial since D^n are contractible.

$$
\begin{tikzcd}
D^n_+ \times \mathbb{R}^k \cong i^*(E) \arrow[r] \arrow[d] & E \arrow[d, "p"] & j^*(E) \cong D^n_- \times \mathbb{R}^k \arrow[l] \arrow[d] \\
D^n_+ \arrow[r, "i"'] & D^n_+ \cup_{S^{n-1}} D^n_- & D^n_- \arrow[l, "j"]
\end{tikzcd}
\tag{4.1}
$$

To obtain the total space E again we need to glue the total spaces of the trivial bundles over the disks. This is done by a function $f : S^{n-1} \times \mathbb{R}^k \to S^{n-1} \times \mathbb{R}^k$ which has the following properties:

- it preserves fibers: $f(x, y) = (x, g(x)(y))$ for $(x, y) \in S^{n-1} \times \mathbb{R}^k$,
- it preserves the vector space structure of the fibers: $g(x) : \mathbb{R}^k \to \mathbb{R}^k$ needs be a linear isomorphism.

The function g can now be succinctly written as $g : S^{n-1} \to \mathrm{GL}(k, \mathbb{R})$ where $\mathrm{GL}(k, \mathbb{R})$ is the group of linear isomorphisms on k-dimensional real vector spaces.

The following is now a vector bundle isomorphic to the original bundle p.

$$
\begin{aligned}
q : D^n_+ \times \mathbb{R}^k \cup_f D^n_- \times \mathbb{R}^k &\to S^n \\
(x, y) &\mapsto x
\end{aligned}
$$

With this construction, one can construct new vector bundles over a sphere. A choice of $g : S^{n-1} \to \mathrm{GL}(k, \mathbb{R})$ completely defines a k-dimensional vector bundle over S^n. Furthermore, up to an isomorphism, this does not depend on the homotopy type of g. Here is a map

$$[S^{n-1}, \mathrm{GL}(k, \mathbb{R})] \to \mathrm{VB}_k(S^n) \tag{4.2}$$

This map is not bijective. It would be bijective if we were to consider complex vector bundles and complex linear isomorphisms. This has to do with the fact that $\mathrm{GL}(k, \mathbb{R})$ is not path connected while $\mathrm{GL}(k, \mathbb{C})$ is.

A very similar result holds for real numbers if one introduces certain orientability conditions (very similar to those defined in 2.3 for spherical fibrations). Namely, the following is a bijection.

$$[S^{n-1}, \mathrm{GL}^+(k, \mathbb{R})] \to \mathrm{VB}^+_k(S^n) \tag{4.3}$$

Here $\mathrm{GL}^+(k, \mathbb{R})$ are matrices with positive determinant and $\mathrm{VB}^+_k(S^n)$ are oriented vector bundles.

It is more suitable to work over $O(k)$ (or $SO(k)$ for oriented case), the group of real, k-dimensional orthogonal linear isomorphism (with positive determinant). This makes certain arguments easier since both $O(k)$ and $SO(k)$ are compact. They can be shown to be homotopy equivalent to $\mathrm{GL}(k, \mathbb{R})$ (or $\mathrm{GL}^+(k, \mathbb{R})$ respectively)[6, p. 26]. We have the following maps

$$[S^{n-1}, O(k)] \to \mathrm{VB}_k(S^n) \tag{4.4}$$

$$([S^{n-1}, SO(k)] \to \mathrm{VB}^+_k(S^n)) \tag{4.5}$$

the latter being a bijection.

4.2 The Clutching Construction for Spherical Bundles

We will define a topological space analogous to $\mathrm{GL}(k, \mathbb{R})$ that is suitable for spherical fibrations.

Definition 4.2.1. [1, p. 139] Let $k \geq 1$. Define a space of homotopy equivalences of a $(k-1)$-sphere $G(k) = \{f : S^{k-1} \to S^{k-1} | \deg f = \pm 1\}$. It is topologized with the compact-open topology.

Likewise define $F(k-1) \subset G(k)$ as such homotopy equivalences which fix a basepoint of S^{k-1}. (Note the shift of the index from k to $k-1$).

Both of the spaces are pointed spaces with basepoint an identity map on S^{k-1}.

The spaces $G(k)$, $F(k-1)$ are topological monoids, which are spaces with a continuous, associative binary operation $\circ : G(k) \times G(k) \to G(k)$. Here the operation is the composition of maps. Homotopies need not be bijective maps, but there is always a homotopy inverse, a map such that both compositions are homotopic to identity. The path components of $G(k)$ are the equivalence classes of homotopy. In other words, two homotopy equivalences $S^{k-1} \to S^{k-1}$ are homotopic if and only if they are in the same path component of $G(k)$.

In vector bundles over an n-sphere, we had the triviality of bundles over the two hemispheres D^n_+, D^n_- of the n-sphere. The same is true for fibrations, any fibration over a contractible space is trivial (Corollary 2.1.15).

Let us redraw the diagram from the previous section for a spherical fibration.

$$\begin{tikzcd}
D^n_+ \times S^{k-1} \cong i^*(E) \arrow[r] \arrow[d] & E \arrow[d, "p"] & j^*(E) \cong D^n_- \times S^{k-1} \arrow[l] \arrow[d] \\
D^n_+ \arrow[r, "i"'] & D^n_+ \cup_{S^{n-1}} D^n_- & D^n_- \arrow[l, "j"']
\end{tikzcd} \tag{4.6}$$

Here we glue the two halves by a map

$$\begin{aligned} f : S^{n-1} \times S^{k-1} &\to S^{n-1} \times S^{k-1} \\ (x, y) &\mapsto (x, g(x)(y)) \end{aligned} \tag{4.7}$$

Such that for each $x \in S^{n-1}$ the map $g(x) : S^{k-1} \to S^{k-1}$ is a homotopy equivalence. In other words g is a map $S^{n-1} \to G(k)$. Just like with vector bundles, the homotopy class of g does not matter and we take the map $[S^{n-1}, G(k)]$. But we may just as well take pointed maps, i.e. elements from $\pi_{n-1}(G(k), \mathrm{id})$. This is because we can change the trivialisation of the lower hemisphere $D^n_- \times S^{k-1}$ to get such a map.

From this we get a total space and a projection

$$\begin{aligned} q : D^n_+ \times S^{k-1} \cup_f D^n_- \times S^{k-1} &\to S^n \\ (x, y) &\mapsto x \end{aligned} \tag{4.8}$$

The proofs of these claims can be found in [1, section 5.2].

And indeed, by the classification Theorem of spherical fibrations due to Stasheff [15] and also [1, Theorem 5.11 and Remark 5.14] we have

Theorem 4.2.2. *The elements of* $\mathrm{SF}_k(S^n)$*, the equivalence classes of* $(k-1)$*-spherical fibrations over* S^n*, are in a one-to-one correspondence with the elements* $\pi_{n-1}(G(k), \mathrm{id})$*.*

Later we will need fibrations with a section. The following is a sufficient condition to have a section.

Lemma 4.2.3. *In such case that we have* $[g] \in \pi_{n-1}(F(k-1))$ *the fibration* $q : X \to S^n$ *of 4.8 has a section (a map* $s : S^n \to X$ *such that* $p \circ s = \mathrm{id}_{S^n}$*).*

Proof. Define $f : S^{n-1} \times S^{k-1} \to S^{n-1} \times S^{k-1}$ and the spherical bundle $q : D^n_+ \times S^{k-1} \cup_f D^n_- \times S^{k-1} \to S^n$ like above

Let $y_0 \in S^{k-1}$ be the basepoint. Then the map

$$\begin{aligned} s : S^n &\to D^n_+ \times S^{k-1} \cup_f D^n_- \times S^{k-1} \\ x &\mapsto (x, y_0) \end{aligned}$$

is well defined and continuous since $f(x, y_0) = (x, g(x)(y_0)) = (x, y_0)$.

□

4.3 The Classifying Spaces

The previous sections described a classification or, in the case of vector bundles, an almost classification result for $\mathrm{VB}_k(S^n)$ and $\mathrm{SF}_k(S^n)$. There are much stronger results which we will summarise in this section.

Definition 4.3.1. [1, Section 5.2 and p. 144]

For $O(k)$ the orthogonal group and for the $G(k)$ the monoid of homotopy equivalences of S^{k-1} there are the following inclusions for all $0 < k$:

$$\begin{aligned} O(k) &\to O(k+1) \\ A &\mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{4.9}$$

$$\begin{aligned} G(k) &\to G(k+1) \\ (S^{k-1} \xrightarrow{\simeq} S^{k-1}) &\xrightarrow{\Sigma} (\Sigma S^{k-1} \xrightarrow{\simeq} \Sigma S^{k-1}) \end{aligned} \tag{4.10}$$

the last map is by the suspension of the homotopy equivalence of S^{k-1}. Suspension of a homotopy equivalence is a homotopy equivalence.

It follows that for a non-negative number l there are the following inclusions $O(k) \to O(k+l)$ and $G(k) \to G(k+l)$ defined by the successive applications of 4.9 and 4.10. Having these maps enables us to take homotopy colimits. According to [1, p. 142, p. 147] the inclusions 4.9 and 4.10 are cofibrations and hence we can take colimits.

$$O = \operatorname{colim}_{k\to\infty} O(k) \tag{4.11}$$

$$G = \operatorname{colim}_{k\to\infty} G(k) \tag{4.12}$$

Theorem 4.3.2. *(Classification of vector bundles)*

1. *(Universal bundle)[12, Theorem 5.23, p. 95] Take any integer $k \geq 0$. Then there exists a so-called classifying space of k-dimensional vector bundles $BO(k)$ and a k-dimensional vector bundle called the universal bundle $EO(k) \to BO(k)$. Let X be a finite CW complex. Then there is one-to-one correspondence between the following sets.*

$$\begin{aligned} [X, BO(k)] &\xrightarrow{bij.} \mathrm{VB}_k(X) \\ f &\mapsto f^*(EO(k)) \end{aligned} \tag{4.13}$$

The correspondence is given by pulling back the universal bundle.

2. *[12, p. 96] The trivial k-dimensional vector bundle $\underline{\mathbb{R}^k}$ over a finite CW complex X is classified by the constant map $X \to BO(k)$, $X \mapsto \{*\}$. The universal bundle $EO(k) \to BO(k)$ is classified by the identity $\mathrm{id}_{BO(k)}$.*

3. *(Pullback)[12, Theorem 5.23, p. 95] Let X and Y be finite CW complexes, $g : Y \to X$ a map, $p : E \to X$ a k-dimensional vector bundle over X classified by the map $f : X \to BO(k)$. Then the pullback of p along g is a vector bundle $g^*(p)$ classified by the map $Y \to BO(k)$ given by the composition $f \circ g : Y \to BO(k)$.*

4. *[1, p.142] The maps $O(k) \to O(k+l)$ induce maps $BO(k) \to BO(k+l)$. They classify the Whitney sum with a trivial l-dimensional vector bundle. In other words if a k-dimensional vector bundle p over a finite CW complex has the classifying map $X \to BO(k)$ then the bundle $p \oplus \underline{\mathbb{R}^l}$ is classified by the composition $X \to BO(k) \to BO(k+l)$.*

5. *(Universal bundle for stable vector bundles) We form the colimit $BO = \mathrm{colim}_{k\to\infty} BO(k)$. For X a finite CW complex there is a one-to-one correspondence between the sets (we saw that $\mathrm{VB}(X)$ is actually a group).*

$$\mathrm{VB}(X) \xrightarrow{bij.} [X, BO] \tag{4.14}$$

The map is defined as follows. Let us take a stable bundle $[p] \in VB(X)$. It has a representative $[p] \in \mathrm{VB}_k(X)$ for some integer k. The vector bundle p is classified by some map $X \to BO(k)$. Define the image of $[p]$ in $[X, BO]$ by post composing it with the map $BO(k) \to BO$. This implies that the following diagram commutes.

$$\begin{tikzcd} \mathrm{VB}_k(X) \arrow[r] \arrow[d] & {[X, BO(k)]} \arrow[d] \\ \mathrm{VB}(X) \arrow[r] & {[X, BO]} \end{tikzcd} \tag{4.15}$$

Let us now state an analogous theorem for the spherical fibrations.

Theorem 4.3.3. *(Classification of spherical fibrations)*

1. *(Universal fibration)[15, Classification Theorem] Take any integer $k \geq 0$. Then there exists a classifying space of $(k-1)$-spherical fibrations $BG(k)$ and a $(k-1)$-spherical fibration $E_{S^{k-1}} \to BG(k)$ called the universal fibration. Let X be a finite CW complex. Then there is a one-to-one correspondence between the following*

sets

$$
\begin{aligned}
[X, BG(k)] &\xrightarrow{bij.} \mathrm{SF}_k(X) \\
f &\mapsto f^*(E_{S^{k-1}})
\end{aligned}
\tag{4.16}
$$

2. *The trivial $(k-1)$-spherical fibration over a finite CW complex X is classified by the constant map $X \to BG(k)$, $X \mapsto \{*\}$. The universal bundle is classified by the identity $\mathrm{id}_{BG(k)}$.*

3. *(Pullback) Let X, Y be finite CW complexes, $g : Y \to X$ a map, p a $(k-1)$-spherical fibration over X classified by the map $f : X \to BG(k)$. Then the pullback of p along g is a $(k-1)$-spherical fibration $g^*(p)$ classified by the map $Y \to BG(k)$ given by the composition $f \circ g : Y \to BG(k)$.*

4. *[1, p. 145] The maps $G(k) \to G(k+l)$ induce maps $BG(k) \to BG(k+l)$. Let $p : E \to X$ be a $(k-1)$-spherical fibration over a finite CW complex X classified by some map $X \to BG(k)$. Then the fiberwise join of p with a trivial $(l-1)$-spherical fibration $p * \underline{S^{l-1}}$ is classified by the composition $X \to BG(k) \to BG(k+l)$.*

5. *We can form a colimit $BG = \mathrm{colim}_{k\to\infty} BG(k)$. For any finite CW complex X there is a one-to-one correspondence between the sets (we saw that $\mathrm{SF}(X)$ is actually a group)*

$$\mathrm{SF}(X) \to [X, BG] \tag{4.17}$$

The map is defined analogously with the vector bundle case. Namely a stable spherical fibration $[p] \in \mathrm{SF}(X)$ is represented by some $(k-1)$-spherical fibration $p \in \mathrm{SF}_k(X)$. Then we obtain the map from $[X, BG]$ by postcomposing the classifying map $X \to BG(k)$ of p with the map $BG(k) \to BG$.

Lemma 4.3.4. *[1, Lemma 5.13, p. 139 and Remark 5.14, p. 140] The space $BG(k)$ is path connected and for any basepoint $x \in BG(k)$ and any $n \geq 1$ we have an isomorphism $\pi_n(BG(k), x) \xrightarrow{\cong} \pi_{n-1}(G(k), id_{S^{n-1}})$. Under this isomorphism an element from $\pi_n(BG(k), x)$ which classifies some fibration over S^n maps to the element from $\pi_n(G(k), \mathrm{id}_{S^{n-1}})$ which is the clutching map for the same spherical fibration.*

The proof is done by having the quasi-fibration $G(k) \to EG(k) \to BG(k)$ with $EG(k)$ contractible. A quasi-fibration is a generalisation of fibration which we will not use further *, but the important property is that there is a long exact sequence of

*A quasi-fibration is a map $f : E \to X$ such that for any $x \in X$, $f^{-1}(x)$ is homotopy equivalent to the homotopy fiber defined in 3.2.1

homotopy groups analogous to that of a fibration. From contractility of $EG(k)$, the Lemma would follow.

Lemma 4.3.5. *[1, p. 147]*

1. *We have*
$$\pi_i(G) = \pi_i(\operatorname{colim}_{k\to\infty} G(k)) = \operatorname{colim}_{k\to\infty} \pi_i(G(k)) \tag{4.18}$$
$$\pi_i(BG) = \pi_i(\operatorname{colim}_{k\to\infty} BG(k)) = \operatorname{colim}_{k\to\infty} \pi_i(BG(k)) \tag{4.19}$$

2. *The isomorphism* $\pi_n(BG(k)) \xrightarrow{\cong} \pi_{n-1}(G(k))$ *from the Lemma 4.3.4 induce an isomorphism on the colimits (by the Lemma 1.4.1)* $\pi_{n+1}(BG) \xrightarrow{\cong} \pi_n(G)$.

Chapter 5

The Exotic Poincaré Complexes

5.1 The Vector Bundle Reduction

Definition 5.1.1. [1, Definition 5.69] Let X be a finite Poincaré complex. The Spivak normal fibration of X *has a vector bundle reduction* if for one and hence all Spivak normal structures (p, c) there exists a vector bundle q over X such that p is stably fiber homotopy equivalent to the underlying spherical fibration Sq in the sense of the definition 2.1.20.

A Poincaré complex without a vector bundle reduction is called an *exotic Poincaré complex.*

Note 5.1.2. A finite Poincaré complex X having a vector bundle reduction is equivalent to the image of the map $S : \mathrm{VB}(X) \to \mathrm{SF}(X)$ (see 2.20) containing $[p]$.

To show that if one Spivak normal structure has the vector bundle reduction all of them do, we need the uniqueness result of 3.3.2. For if we have two Spivak normal fibrations p_1 and p_2, then they are stably fiber homotopy equivalent, in other words $[p_1] = [p_2]$ in $\mathrm{SF}(X)$. Then if there is a vector bundle q over X such that Sq is stably fiber homotopy equivalent to p_1, i.e. $[p_1] = [Sq]$. Then also $[p_2] = [Sq]$ in $\mathrm{SF}(X)$.

The following discussion is from [1, pp. 163-165]. We will explain some of the steps in more detail.

We would like to have some equivalent conditions for a finite Poincaré complex to have a vector bundle reduction of its Spivak normal fibration.

Take $EO(k) \to BO(k)$ is the k-dimensional universal vector bundle defined in the Theorem 4.3.2. This vector bundle has an associated $(k-1)$-spherical fibration $S(EO(k)) \to BO(k)$. Define a map $J_k : BO(k) \to BG(k)$ as the classifying map of this spherical fibration. Taking the homotopy colimit $\mathrm{hocolim}_{k\to\infty} J_k = J'$ yields us a map $J' : \mathrm{hocolim}_{K\to\infty} BO(k) \to \mathrm{hocolim}_{K\to\infty} BG(k)$. Those spaces are homotopy equivalent to spaces BO and BG respectively so we obtain the map

$$J : BO \to BG \tag{5.1}$$

Take any finite CW complex X. Note that J_k and J induce maps $[X, BO(k)] \to [X, BG(k)]$ and $[X, BO] \to [X, BG]$ respectively. Under the bijections of Theorems 4.3.2 and 4.3.3 these maps correspond to maps $S : \mathrm{VB}_k(X) \to \mathrm{SF}_k(X)$ and $S : \mathrm{VB}(X) \to \mathrm{SF}(X)$ respectively.

It turns out ([12, Proposition 9.20, p. 201] and [1, p. 164]) that the homotopy fiber of the map $BO \to BG$ is a certain space G/O and there is a space $B(G/O)$ and a map $BG \to B(G/O)$ for which $BO \xrightarrow{J} BG \to B(G/O)$ is a so-called homotopy fibration sequence* and on homotopy groups, we have for $i \geq 0$

$$\pi_{i+1}(B(G/O)) \cong \pi_i(G/O) \tag{5.2}$$

Take a compact n-dimensional manifold M. It has an n-dimensional tangent bundle $\mathrm{TM} \to M$. By the classification Theorem 4.3.2 it defines a unique element $[t_{M,n}] \in [M, BO(n)]$. We can stabilise this map by postcomposition with $BO(n) \to BO$ obtaining the classifying map of the stable tangent bundle

$$[t_M] \in [M, BO] \tag{5.3}$$

If we then choose an embedding $i : M \to \mathbb{R}^{n+k}$ we have a normal vector bundle $[\nu(i)] \in \mathrm{VB}_k(X)$. As we explained before, stabilisation of the normal bundle corresponds to embedding M into a larger Euclidean spaces by postcomposing i with the maps $\mathbb{R}^{n+k} \to \mathbb{R}^{n+k+l}$. Denote the stable normal bundle by

$$[\nu_M] \in \mathrm{VB}(X) \tag{5.4}$$

As we have seen in 2.2.2 it is independent on the choice of i and is an inverse of the tangent bundle. Under the identifications in the classification Theorems 4.3.2, 4.3.3 we can write $[t_M] = -[\nu_M]$.

Now take a Poincaré complex X and any Spivak normal $(k-1)$-fibration p. From the stabilisation of $[p] \in \mathrm{SF}_k(X)$ we get a stable fibration

*The definition of a homotopy fibration sequence is in the language of model categories, which exceeds the scope of this thesis. The important fact is that, just as with fibrations, there is a long exact sequence of homotopy groups. Furthermore, we also have the lemma 2.1.5

$$[\nu_X] \in \mathrm{SF}(X) \tag{5.5}$$

independent of the choice of p, by the uniqueness of the Spivak normal fibration 3.3.2. To p there corresponds an element $[s_{X,k}] \in [X, BG(k)]$. By a postcomposition with $BG(k) \to BG$ we obtain an element

$$[s_X] \in [X, BG] \tag{5.6}$$

This is actually the element which corresponds to $[\nu_X]$ under the map 4.17 of the Classification theorem 4.3.3.

A compact manifold M, considered as a Poincaré complex has a Spivak normal structure constructed in Lemma 3.1.1, where its Spivak normal fibration p was constructed as the associated spherical fibration to a normal bundle $\nu(i)$ of M. In the terms just defined, we have

$$\begin{gathered} J_{k*} : [X, BO(k)] \to [X, BG(k)] \\ [\nu(i)] \mapsto [s_{M,k}] \end{gathered} \tag{5.7}$$

and the same for the stabilisations

$$\begin{gathered} J_* : [X, BO] \to [X, BG] \\ [\nu_M] \mapsto [s_M] \end{gathered} \tag{5.8}$$

The obstruction to a Poincaré complex X being homotopy equivalent to a manifold will be that the map J_* has $[s_X]$ in its image. For that we need, that homotopy equivalences behave well with the elements $[\nu_X]$ and $[s_X]$.

Lemma 5.1.3. *Let $f : X \to Y$ be a homotopy equivalence of finite Poincaré complexes. Then for the elements $[\nu_X]$, $[\nu_Y]$ and $[s_X]$, $[s_Y]$ defined in 5.5 and 5.6 the map f behaves in the following way*

$$\begin{gathered} f^* : \mathrm{SF}(Y) \to \mathrm{SF}(X) \\ [\nu_Y] \mapsto [\nu_X] \end{gathered} \tag{5.9}$$

where f^* is the pullback along f

and

$$f^* : [Y, BG] \to [X, BG] \qquad (5.10)$$
$$[s_Y] \mapsto [s_X]$$

where f^* is a precomposition by f.

Proof. The second claim follows from the first by the classification Theorem of spherical fibrations 4.3.3.

Let us take the map $f^* : \mathrm{SF}(Y) \to \mathrm{SF}(X)$ and (ν_Y, c_Y) a Spivak normal structure of Y. For the spherical fibration $\nu_X = f^*(\nu_Y)$ to be a Spivak normal fibration of X we need an element $c_X : S^{n+k} \to \mathrm{Th}(\nu_x)$ such that (ν_X, c_X) is the Spivak normal structure of X, namely that the Hurewicz homomorphism maps c to a generator of $H_{n+k}(\mathrm{Th}(\nu_X))$.

The pullback $\nu_X = f^*(\nu_Y)$ induces the map of total spaces $E\nu_X \to E\nu_Y$ of the fibrations ν_X and ν_Y. Since f is a homotopy equivalence, so is this map. A map of the total spaces induces the map of Thom spaces of the fibrations. Thus we have a homotopy equivalence

$$\mathrm{Th}(f) : \mathrm{Th}(\nu_X) \xrightarrow{\simeq} \mathrm{Th}(\nu_Y) \qquad (5.11)$$

Let us define c_X as the composition $S^{n+k} \xrightarrow{c_Y} \mathrm{Th}(\nu_Y) \xrightarrow{\mathrm{Th}(f)^{-1}} \mathrm{Th}(\nu_X)$ where the map $\mathrm{Th}(f)^{-1}$ is the homotopy inverse of $\mathrm{Th}(f)$.

The following diagram commutes

$$\begin{tikzcd}
\pi_{n+k}(\mathrm{Th}(\nu_X)) \arrow[r, "h"] \arrow[d, "\mathrm{Th}(f)_*"] & H_{n+k}(\mathrm{Th}(\nu_X)) \arrow[d, "\mathrm{Th}(f)_*"] \\
\pi_{n+k}(\mathrm{Th}(\nu_Y)) \arrow[r, "h"] & H_{n+k}(\mathrm{Th}(\nu_Y))
\end{tikzcd} \qquad (5.12)$$

since for any element $c \in \pi_{n+k}(\mathrm{Th}(\nu_X))$ we have

$$h(\mathrm{Th}(f)_*(c)) = h((\mathrm{Th}(f) \circ c)) = (\mathrm{Th}(f) \circ c)_*([S^{n+k}]) =$$
$$(\mathrm{Th}(f)_* \circ c_*)([S^{n+k}]) = \mathrm{Th}(f)_* \circ (c_*([S^{n+k}])) = \mathrm{Th}(f)_* \circ h(c) \qquad (5.13)$$

Since $\mathrm{Th}(f)$ is a homotopy equivalence, the vertical maps are isomorphisms. By definition the left vertical map sends c_x to c_y. Since c_y is part of the Spivak normal structure we get that the lower horizontal map sends c_Y to the generator of $H_{n+k}(\mathrm{Th}(\nu_Y))$. Hence

c_X also gets sent to a generator of $H_{n+k}(\mathrm{Th}(\nu_X))$. $\square$

Lemma 5.1.4. *[1, Lemma 5.78, p. 165.] Let X be a finite Poincaré complex homotopy equivalent to some closed manifold. Then the following equivalent statements hold.*

i) Any Spivak normal fibration of X has a vector bundle reduction of the Definition 5.1.1.

ii) The element $[\nu_X] \in \mathrm{SF}(X)$ lies in the image of the map $S : \mathrm{VB}(X) \to \mathrm{SF}(X)$.

iii) The element $[s_X] \in [X, BG]$ lies in the image of the map $J_* : [X, BO] \to [X, BG]$ induced by the map $J : BO \to BG$.

iv) The element $[s_X] \in [X, BG]$ is sent to a class of a constant map under the map $q_* : [X, BG] \to [X, B(G/O)]$.

Proof. First, let us do the equivalence of the statements $i) - iv)$

The proof that conditions $i), ii), iii)$ are equivalent is exactly the discussion above.

The conditions $iii)$ and $iv)$ are equivalent by the obstruction property of fibrations from Lemma 2.1.6.

Now assume that X is homotopy equivalent to some closed manifold M by some homotopy equivalence $f : X \to M$. We will prove the condition $ii)$

Let us consider the diagram on the left. It commutes

$$\begin{tikzcd} \mathrm{VB}(M) \arrow[r, "f^*"] \arrow[d, "S"] & \mathrm{VB}(X) \arrow[d, "S"] & {[\nu_M]} \arrow[d, mapsto] & \\ \mathrm{SF}(M) \arrow[r, "f^*"] & \mathrm{SF}(X) & {[S\nu_M]} \arrow[r, mapsto] & {[\nu_X]} \end{tikzcd} \tag{5.14}$$

where the horizontal maps are induced by f and the vertical maps are the maps 2.20. Let us have the stable normal bundle $[\nu_M]$ of M. Then $[S\nu_M]$ is the stabilisation of the Spivak normal fibration of M. Since f is a homotopy equivalence, we have by 5.9 that $[S\nu_M]$ gets mapped to $[\nu_X]$ the Spivak normal fibration of X. Now by commutativity of 5.14 we have that $[\nu_X]$ is in the image of S. $\square$

5.2 The Space X^5

First we state without proof the following useful Lemma.

Lemma 5.2.1. *[1, Lemma 5.82, p. 166] Let $p : E \to X$ be a $(k-1)$-spherical fibration over a finite Poincaré complex X. Then for the stable Spivak normal fibrations $[\nu_E]$ and $[\nu_X]$ for E and X respectively we have*

$$[\nu_E] = p^*([\nu_X]) - p^*([p]) \tag{5.15}$$

The last term $p^*([p])$ is just the fibration p pullbacked along its own map p.

This could be viewed as an analogy of a certain property of vector bundles. Let $p : E \to X$ be a vector bundle. Then the tangent bundle $T_X E$ of E restricted to X is isomorphic to $p \oplus TM$ (see [7, Theorem 2.1, p. 94]). Moreover, if we have the associated spherical fibration $Sp : SE \to X$, one can show that there is an isomorphism of vector bundles over SE

$$T(SE) \oplus \underline{\mathbb{R}} \xleftarrow{\cong} (Sp)^* TX \oplus (Sp)^*(p) \tag{5.16}$$

We know that normal bundles are stable inverses to tangent bundles. Hence it can be shown that in $\mathrm{VB}(SE)$ we have

$$-[\nu_{SE}] = -(Sp)^*([\nu_X]) + (Sp)^*(p) \tag{5.17}$$

The proof of the previous claims can be found in [1, p. 165]

The following table is from [12, p. 203]. We will need some of these groups in what follows.

n	1	2	3	4	5	6	7	8	9	10
$\pi_n(G/O)$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z} \oplus \mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
$\pi_n(BO)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\pi_n(BG)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$

Table 5.1: Low dimensional homotopy groups of some spaces.

The following construction is from [1, Example 5.85]. We have added a lot of details.

The goal of the following section is to show that in $[S^3, BG(3)]$ there is a map that classifies a 2-spherical fibration over a 3-sphere that does not have a vector bundle reduction.

Since, by the table 5.1, $\pi_3(BO)$ vanishes, we have, by the long exact sequence of homotopy groups of the homotopy fibration sequence $BO \xrightarrow{J} BG \xrightarrow{q} B(G/O)$ that $\pi_3(q) : \pi_3(BG) \to \pi_3(B(G/O))$ is injective. Since both of the groups are $\mathbb{Z}/2$, the map is also bijective and hence an isomorphism. We would like to understand this in terms of unpointed maps. From the Lemma 2.1.5 We have that

$$[S^3, BO] \to [S^3, BG] \xrightarrow{q_*} [S^3, B(G/O)]$$

is exact at $[S^3, BG]$. But from the triviality of $\pi_3(BO)$ we have that $[S^3, BO]$ is also trivial, because any homotopy relative the basepoint of S^3 is also a free homotopy. Thus q_* is injective. It is also surjective. For take an element $f : S^3 \to B(G/O) \in [S^3, B(G/O)]$, basepoints $s_0 \in S^3$, $x_0 = f(s_0)$, any $e_0 \in q^{-1}(x_0) \subset BG$. From the long exact sequence of homotopy groups of the fibration q with these basepoints we have $f \in \pi_3(B(G/O), x_0)$, but this has a preimage g in $\pi_3(BG, e_0)$ since $q_* : \pi_3(BG, e_0) \to \pi_3(B(G/O), x_0)$ is an isomorphism. The element g is also in the set $[S^3, BG]$ and it is a preimage of f.

Thus $[S^3, BG] \xrightarrow{q_*} [S^3, B(G/O)]$ is a bijection. Note that those sets are not necessarily two-element sets, as the homotopy groups are. A nontrivial element of $[S^3, BG]$ classifies some stable spherical fibration. The total space of this fibration will be some exotic Poincaré complex without vector bundle reduction. The challenges with this approach are the following

- An element from $[S^3, BG]$ classifies some stable spherical fibration $[p] \in \mathrm{SF}(X)$. We would like to know the smallest representative of $[p]$, i.e. the representative with fiber of the least dimension. It will turn out that the smallest such fibration has a 2-dimensional fiber $S^2 \to X \xrightarrow{p} S^3$.

- To show that the total space does not have a vector bundle reduction we need to show that the image of the element $[s_X] \in [X, BG]$ classifying the Spivak normal fibration of X under the map $[X, BG] \xrightarrow{q_*} [X, B(G/O)]$ is non-trivial. For the 2-spherical fibration mentioned previously, we are not aware of how to accomplish this without proving additionally proving that our fibration p has a section.

A $(k-1)$-spherical fibration over S^n, in addition to being classified by a map $[S^n, BG(k)]$ is also classified by the map $\pi_{n-1}(G(k))$ by the means of the clutching construction. We know that such a fibration has a section if its clutching map is actually from $\pi_{n-1}(F(k-1))$ (Recall that $F(k-1)$ is the space of pointed homotopy equivalences of S^{k-1} and recall the Lemma 4.2.3).

Now we want to justify a commutative diagram

$$\begin{tikzcd} \pi_2(F(2)) \arrow[r] \arrow[d, "\cong"] & \pi_2(G) \arrow[d, "\cong"] \\ \pi_4(S^2) \arrow[r, "\cong"] & \pi_2^s \end{tikzcd} \tag{5.18}$$

for the top horizontal map induced by $F(2) \to G(3) \to G$.

Assume we have such a diagram 5.18. A nontrivial element in $[S^3, BG]$ is by the Lemma 4.3.5 in $\pi_2(G)$. Since three maps in the diagram are isomorphisms, so is the fourth map $\pi_2(F) \to \pi_2(G(3)) \to \pi_2(G)$. The nontrivial element in $h \in \pi_2(G)$ simultaneously comes from $\pi_2(G(3))$ and hence classifies a fibration $S^2 \to X \xrightarrow{p} S^3$ and also comes from $\pi_2(F(2))$ and thus the fibration p has a section.

Calculation 5.2.2. (Proof of the diagram 5.18 part 1.) Recall that for any pointed space X, the *loop space* ΩX is the space $\langle S^1, X\rangle$ (the space of pointed maps $S^1 \to X$, with basepoint the constant loop). We can have iterated loop spaces $\Omega^k X$ defined inductively $\Omega^k X = \Omega(\Omega^{k-1} X)$.

By [5, p. 395] we have for $i \geq 0$

$$\pi_{i+1}(X) \cong \pi_i(\Omega X) \tag{5.19}$$

We want to observe that $\Omega^k X$ is homotopy equivalent to $\left\langle S^k, X\right\rangle$. In general, from a pointed map $X \to \langle Y, Z\rangle$ one has an obvious map $X \times Y \to Z$. But one can do better. Since this map is constant on $X \times \{y_0\} \vee \{x_0\} \times Y$ we actually have a map $X \times Y / X \vee Y \cong X \wedge Y \to Z$. From this $\Omega^2 X \cong \langle S^1 \wedge S^1, X\rangle$. Since we have $S^k \wedge S^l \simeq S^{k+l}$ the result follows by induction.

Take the k-th loop space of a k-sphere $\Omega^k S^k$. By the previous observation it is the $\left\langle S^k, S^k\right\rangle$. Maps from a sphere to itself are homotopy equivalent if and only if they have the same degree [5, Section 2.2, p.134]. For any integer d there is a pointed map $S^k \to S^k$ of degree $d^\dagger$. Hence there are $\mathbb{Z}$ path components of $\left\langle S^k, S^k\right\rangle$. By definition $F(k) \subset \Omega^k S^k$ and actually $F(k)$ consists of two path components of $\Omega^k S^k$, those with degrees ± 1.

Since homotopy groups π_i for $i \geq 1$ do not see path components other than the one with the basepoint we have for $i \geq 1$

$$\pi_i(F(k)) = \pi_i(\Omega^k S^k, \mathrm{id}_{S^K}) \cong \pi_i(\Omega^k S^k, *) \cong \pi_{i+k}(S^k) \tag{5.20}$$

The last equality is by the equation 5.19. The middle equality is by a more technical property. The space ΩX actually has an additional structure: a binary operation (defined by the composition of loops), which makes it a so-called H-space. In such a space all path components have the same homotopy type.

†The following is a well known construction. Take a number d. The map $f_d : S^1 \to S^1$ defined by $e^{i\lambda} \mapsto e^{id\lambda}$ is of degree d. There is a property of degrees that they are preserved under a suspension ($\deg(f) = \deg(\Sigma f)$). All the maps $\Sigma^{k-1} f : S^k \to S^k$ are of degree d

Calculation 5.2.3. (Proof of the diagram 5.18 part 2.) The lower horizontal map in the diagram 5.18 is the colimit map of the suspensions $\pi_4(S^2) \to \pi_5(S^3) \to \cdots$. By the Section 1.5 all these maps and hence the map $\pi_4(S^2) \to \pi_2^s$ are isomorphisms.

Calculation 5.2.4. (Proof of the diagram 5.18 part 3.) Colimits of homotopy groups commute with homotopy colimits of spaces, see 1.4.1.

By the last discussion and 1.4.1 we have

$$\operatorname{colim}_{k\to\infty} \pi_i(F(k)) = \pi_i(F) \xrightarrow{\cong} \operatorname{colim}_{k\to\infty} \pi_{i+k}(S^k) = \pi_i^s \tag{5.21}$$

Recall that we are actually after the second homotopy group of G. But it turns out, that that $\pi_i(G) \cong \pi_i(F)$. This is true because the following is a fibration [10, Lemma 3.1, p. 46]

$$F(k-1) \to G(k) \xrightarrow{\text{ev.}} S^k \tag{5.22}$$

The first map is the inclusion, second map is the evaluation map $G(k) \to S^{k-1}$ which sends $f \in G(k)$ to $f(y_0)$ where y_0 is the basepoint of S^{k-1}. The fiber is correct since $(\text{ev.})^{-1}(y_0)$ are exactly the maps $S^{k-1} \to S^{k-1}$ which fix the basepoint. In other words, maps from $F(k-1)$.

For any integer i and any $k > i+2$ we have $\pi_i(F(k-1)) \cong \pi_i(G(k))$, since we only have to look at the the appropriate part of the long exact sequence of homotopy groups for the fibration and observe that $\pi_i(S^{k-1}) = \pi_{i+1}(S^{k-1}) = 0$ for large enough k.

This gives us an isomorphism $\pi_i(G(k)) \xrightarrow{\cong} \pi_{i+k-1}(S^{k-1})$, for $k > i+2$. From the Lemma 1.4.1 this suffices to give us an isomorphism $\pi_i(G) \xrightarrow{\cong} \pi_i^s$.

In our case $i = 2$, so the smallest admissible is $k = 5$.

We add some groups into the diagram 5.18

$$\begin{tikzcd}
\pi_2(F(2)) \arrow[r] \arrow[d, "\cong"] & \pi_2(F(4)) \cong \pi_2(G(5)) \arrow[r] \arrow[d, "\cong"] & \pi_2(G) \arrow[d, "\cong"] \\
\pi_4(S^2) \arrow[r, "\cong"] & \pi_6(S^4) \arrow[r, "\cong"] & \pi_2^s
\end{tikzcd} \tag{5.23}$$

We have already proven all the marked isomorphism. The right square commutes by the colimit property in Lemma 1.4.1. In this square, we get that the top horizontal map $\pi_2(G(5)) \to \pi_2(S)$ is an isomorphism.

The left square commutes, so the upper horizontal map is also an isomorphism.

This finishes the proof of the commutativity of the diagram.

We will now show that the total space X of the fibration p is a Poincaré complex, but not homotopy equivalent to a manifold, making use of the necessary conditions in Lemma 5.1.4.

It is a Theorem of Gottlieb [4, Theorem 1, p. 148] that the total space of a fibration with base space and a fiber a Poincaré complex is again a Poincaré complex. Thus X is a Poincaré complex.

By the Lemma 5.2.1 we know that the stable Spivak normal fibration of X is $[\nu_X] = p^*([\nu_{S^3}]) - p^*([p])$. The normal vector bundle of any sphere is trivial since a k-sphere can be embedded is a $(k+1)$-Euclidean space $i : S^k \to \mathbb{R}^{k+1}$, but this 1-dimensional normal vector bundle has an everywhere non-zero section, namely an outward unit vector. So also its Spivak normal fibration is a trivial fibration. Now we get $[\nu_X] = -p^*([p])$.

Let $f_p : S^3 \to BG(3) \to BG$ be the characteristic map of the fibration p composed with the inclusion. Since the stable Spivak normal fibration of X is the pullback of $-[p]$ along the map p, from the Classification Theorem of spherical fibrations 4.3.3 we get that the classifying map of $-[\nu_X]$ is given by $\widetilde{f} = f_p \circ p$.

This is our situation

$$\begin{tikzcd}
X \arrow[d, "p"] \arrow[dr, "\widetilde{f}"] & BO \arrow[d] \\
S^3 \arrow[u, bend left, "s"] \arrow[r, "f_p"] & BG \arrow[d, "q"] \\
& B(G/O)
\end{tikzcd} \tag{5.24}$$

The previous diagram commutes in the sense that $f_p \circ p = \widetilde{f}$ and $\widetilde{f} \circ s = f_p$. The latter is because since s is a section we have $\widetilde{f} \circ s = f_p \circ p \circ s = f_p \circ \mathrm{id} = f_p$. It is *not* necessarily true that $s \circ p = \mathrm{id}$.

To use the Lemma 5.1.4, namely the condition *iv)* we need to show that $\widetilde{f} \in [X, BG]$ is sent to the non-trivial homotopy class by the induced mapping $q_* : [X, BG] \to [X, B(G/O)]$. To do this, we will show that $\pi_3(q \circ \widetilde{f}) : \pi_3(X) \to \pi_3(B(G/O))$ is nontrivial. Let us apply the functor $\pi_3(-)$ to the diagram 5.24

$$\begin{tikzcd}
\pi_3(X) \arrow[d, "\pi_3(p)"] \arrow[dr, "\pi_3(\widetilde{f})"] & 0 \arrow[d] \\
Z \arrow[u, bend left, "\pi_3(s)"] \arrow[r, "\pi_3(f_p)"] & \mathbb{Z}/2 \arrow[d, "q_*"', "\cong"] \\
& \mathbb{Z}/2
\end{tikzcd} \tag{5.25}$$

The map induced by f_p is nontrivial

$$\begin{aligned} \pi_3(f_p) : \pi_3(S^3) &\to \pi_3(BG) \\ \mathbb{Z} &\to \mathbb{Z}/2 \\ 1 &\mapsto 1 \end{aligned} \tag{5.26}$$

since f_p is nontrivial in $[S^3, BG]$.

Now the section of p will become useful, since $\pi_3(\widetilde{f}) \circ \pi_3(s)$ is surjective and so is $\pi_3(q_*) \circ \pi_3(\widetilde{f}) \circ \pi_3(s)$. From this it follows that $q \circ \widetilde{f}$ is not nullhomotopic and so X is a Poincaré complex without a vector bundle reduction and hence not homotopy equivalent to a manifold.

Recall that X was constructed from an element from $\pi_2(G(3))$ by a clutching construction and so can be written as $X^5 = D^3_+ \times S^2 \cup_f D^3_- \times S^2$ (written with the superscript to indicate the dimension of the Poincaré complex) for the map

$$\begin{aligned} f : \partial D^3_+ \times S^2 &\to \partial D^3_- \times S^2 \\ (x, y) &\mapsto (x, h(x)(y)) \end{aligned} \tag{5.27}$$

where h is the nontrivial element of $\pi_2(F(2))$.

5.3 Some Homotopy Groups

Consider the k-sphere S^k as a CW complex composed of one 0-cell and one k-cell.

Let $S^k \vee S^l$ be the wedge of a k-sphere and an l-sphere. Its CW structure is such that the k-cell and the l-cell are attached with the constant attaching map. Let, throughout this chapter, ι_k and ι_l be the pointed inclusion maps $S^k \hookrightarrow S^k \vee S^l$ and $S^l \hookrightarrow S^k \vee S^l$ respectively.

Consider the space $S^k \times S^l$. It can be written as a 0-cell, a k-cell, an l-cell and a $(k+l)$-cell 1.1.4. The k-cell is attached by the constant map and so is the l-cell. Hence there is $S^k \vee S^l$ in $S^k \times S^l$ and the $(k+l)$-cell is attached to it.

Definition 5.3.1. [18, p. 472] or [5, p. 381] Let $k, l > 0$. Let X be a space and take $\alpha \in \pi_p(X), \beta \in \pi_q(X)$. Then the *Whitehead product* $[f, g] \in \pi_{p+q-1}(X)$ is the composition $S^{k+l-1} \to S^k \vee S^l \xrightarrow{f \vee g} X$, where the first map is the attaching map of the (k+l)-cell of $S^k \times S^l$ described above.

For example the attaching map of the $(k+l)$-cell in $S^k \times S^l$ as the Whitehead product $[\iota_k, \iota_l]$.

Calculation 5.3.2. We will need to know the group $\pi_4(S^2 \vee S^3)$. We will follow the method from [5, Example 4.52, p. 380]. First, let's consider a portion of the long exact sequence of homotopy groups for a pair $(S^2 \times S^3, S^2 \vee S^3)$

$$\begin{gathered}\cdots \to \pi_5(S^2 \vee S^3) \to \pi_5(S^2 \times S^3) \to \pi_5(S^2 \times S^3, S^2 \vee S^3) \xrightarrow{\partial} \pi_4(S^2 \vee S^3) \to \\ \longrightarrow \pi_4(S^2 \times S^3) \to \pi_4(S^2 \times S^3, S^2 \vee S^3) \longrightarrow \cdots\end{gathered} \tag{5.28}$$

For any collection $\{X_\alpha\}$ of path connected spaces we have for all $n > 0$ an isomorphism $\pi_n(\prod_\alpha X_\alpha) \cong \prod_\alpha \pi_n(X_\alpha)$ defined by sending a map $\varphi : S^n \to \prod_\alpha X_\alpha$ to an element in the target whose β's component is $p_{\beta*} \circ \varphi$ where p_β is the projection $\prod_\beta X_\alpha \to X_\beta$ [5, Prop. 4.2, p. 343]. Consider the following maps from the long exact sequence $\pi_i(S^2 \vee S^3) \to \pi_i(S^2 \times S^3) \cong \pi_i(S^2) \oplus \pi_i(S^3)$ for $i = 4, 5$. They are induced by inclusion of spaces. Consider the group $\pi_i(S^2) \oplus \pi_i(S^3)$. It is generated by maps $S^i \to S^2$ and $S^i \to S^3$. Any such map has a preimage in $\pi_i(S^2 \vee S^3)$ by postcomposing it with an inclusion ι_2 or ι_3 to get a map $S^i \to S^2 \xrightarrow{\iota_2} S^2 \vee S^3$ or $S^i \to S^3 \xrightarrow{\iota_3} S^2 \vee S^3$.

Surjectivity gives us the following information about the maps

$$\begin{gathered}\cdots \to \pi_5(S^2 \vee S^3) \twoheadrightarrow \pi_5(S^2 \times S^3) \xrightarrow{0} \pi_5(S^2 \times S^3, S^2 \vee S^3) \xhookrightarrow{\partial} \pi_4(S^2 \vee S^3) \twoheadrightarrow \\ \longrightarrow\!\!\!\!\!\twoheadrightarrow \pi_4(S^2 \times S^3) \xrightarrow{0} \pi_4(S^2 \times S^3, S^2 \vee S^3) \longrightarrow \cdots\end{gathered} \tag{5.29}$$

Hence the long exact sequence yields a short exact sequence

$$0 \to \pi_5(S^2 \times S^3, S^2 \vee S^3) \xrightarrow{\partial} \pi_4(S^2 \vee S^3) \xrightarrow{i_*} \pi_4(S^2 \times S^3) \to 0 \tag{5.30}$$

The map i_*, upon identifying its target with $\pi_4(S^2) \oplus \pi_4(S^3)$, sends the class of a map map $\varphi : S^4 \to S^2 \vee S^3$ to the pair $(p_2 \circ \varphi, p_3 \circ \varphi)$ for projections $p_2 : S^2 \vee S^3 \to S^2$ and $p_3 : S^2 \vee S^3 \to S^3$.

The sequence 5.30 splits as there is a section s.

$$s : \pi_4(S^2 \times S^3) \cong \pi_4(S^2) \oplus \pi_4(S^3) \to \pi_4(S^2 \vee S^3) \tag{5.31}$$

Defined by $s(\phi) = (\iota_2)_* \circ \phi$ for $\phi \in \pi_4(S^2)$, $s(\psi) = (\iota_3)_* \circ \phi$ for $\psi \in \pi_4(S^3)$ and extended linearly.

The splitting gives

$$\begin{aligned}\pi_4(S^2 \vee S^3) &\cong s(\pi_4(S^2 \times S^3)) \oplus \partial(\pi_5(S^2 \times S^3, S^2 \vee S^3)) \cong \\ &\cong \pi_4(S^2) \oplus \pi_4(S^3) \oplus \pi_5(S^2 \times S^3, S^2 \vee S^3)\end{aligned} \tag{5.32}$$

Let us now study the summands. In the Section 1.5 we saw that $\pi_3(S^4) \cong \mathbb{Z}/2$ with generator $\sigma\eta$ the suspension of the Hopf map $\eta : S^3 \to S^2$. From the same section, we have $\pi_4(S^2) \cong \mathbb{Z}/2$ and it is generated by $\eta \circ \sigma\eta$. We will abbreviate this as η^2.

The following Lemma will describe the last summand $\pi_5(S^2 \times S^3, S^2 \vee S^3)$. It is proven by a technique in [5, pp. 380-381].

Lemma 5.3.3. *The group $\pi_5(S^2 \times S^3, S^2 \vee S^3)$ is infinite cyclic and generated by the characteristic map of the 5-cell attached to $S^2 \vee S^3$ by the Whitehead product $[\iota_2, \iota_3]$.*

Proof. If we could use the homotopy excision Theorem 1.1.6 to get $\pi_5(S^2 \times S^3, S^2 \vee S^3) \cong \pi_5(S^2 \times S^3/S^2 \vee S^3)$ we would be finished since $S^2 \times S^3/S^2 \vee S^3$ is homeomorphic to S^5 and $\pi_5(S^5) \cong \mathbb{Z}$.

The excision Theorem for homotopy, in this case, requires there exist r and s, $5 \leq r + s$, such that $(S^2 \times S^3, S^2 \vee S^3)$ is r-connected and $S^2 \vee S^3$ is s-connected.

Constructing a space with a minimal CW structure (as few cells as possible) is advantageous in showing the high connectedness of cellular maps or CW pairs. See [5, Corollary 4.12 and pp. 352-357]. In particular, by [5, Corollary 4.12] the pair $(S^2 \times S^3, S^2 \vee S^3)$ is 4-connected since the wedge $S^2 \vee S^3$ is the 4-skeleton of $S^2 \times S^3$.

The space $S^2 \vee S^3$ is 1-connected since it is connected and does not have any 1-cells.

Therefore we can use the Excision theorem 1.1.6 and we have $\pi_5(S^2 \times S^3, S^2 \vee S^3) \cong \mathbb{Z}$

To find the generator we use the isomorphism $\pi_5(S^2 \times S^3, S^2 \vee S^3) \xrightarrow{\cong} \pi_5(S^5)$. The generator of the right-hand side is the characteristic map of the 5-cell. Its preimage is represented by the characteristic map of the 5-cell in $S^2 \times S^3$. This cell is attached by the Whitehead product by the definition of the product. □

5.4 Comparing X^5 and Y^5

In the book *The classifying spaces for surgery and cobordism of manifolds* [10] Madsen and Milgram define the space

$$Y^5 = D^5 \cup_h \left(S^2 \vee S^3\right) \tag{5.33}$$

with $h = ([\iota_2, \iota_3] + \iota_2 \circ \eta^2) : S^4 \to S^2 \vee S^3$, where $[\iota_2, \iota_3]$ is the Whitehead product of inclusions and η^2 is the generator of $\pi_4(S^2)$.

The remainder of this chapter used to prove the following.

Theorem 5.4.1. *The space X^5 is homotopy equivalent to Y^5.*

Firstly we want to show that X^5 can be written as $D^5 \cup (S^2 \vee S^3)$ for some attaching map.

The space X^5 is characterised in 5.27 as a gluing of $D^3_+ \times S^2$ to $D^3_- \times S^2$. We will now collapse one of the disks fiber-wise. More precisely let $\sim$ be an equivalence relation on $D^3_- \times S^2$ such that $(x, y) \sim (x', y)$ for $x, x' \in D^3_-$ and $y \in S^2$. We factor X^5 by $\sim$ obtaining a homotopy equivalent space $X'^5 \simeq X = X^5 / \sim$.

Recall that in the equation 5.27 the map f was defined using $h \in \pi_2(F(2))$. We can use it to characterise the space X'^5

$$\begin{aligned} X'^5 &= D^3 \times S^2 \cup_{f'} S^2 \\ f' &: \partial D^3 \times S^2 \to S^2 \\ &(x, y) \mapsto h(x)(y) \end{aligned} \tag{5.34}$$

We want to show the following

Lemma 5.4.2. *The space X'^5 is homotopy equivalent to some space $X''^5 = D^5 \cup_{f''} S^2 \vee S^3$ for some attaching map $f'' \in \pi_4(S^2 \vee S^3)$.*

Proof. Let us first find a suitable embedding of $S^2 \vee S^3$ in X'^5.

Let y_0 be the basepoint of S^2. Since h is from the group $\pi_2(F(2))$ we have $h(x)(y_0) = y_0$. Notice that in X'^5 the subspace $D^3 \times \{y_0\} \subset D^3 \times S^2$ is actually an S^3, since the boundary of $D^3 \times \{y_0\}$ gets collapsed to a point precisely because $h(x)(y_0) = y_0$. Let the $S^2 \subset D^3 \times S^2 \cup_{f'} S^2$ be inclusion into the sphere on the right of the union. Combining these two subspaces yields an inclusion $S^3 \vee S^2 \to X'^5$, because they intersect at exactly one point y_0.

We need to show that when $S^2 \vee S^3$ is removed from X'^5, we are left with a 5-disk. We will remove the spheres one by one for clarity.

$$(D^3 \times S^2 \cup_{f'} S^2) \setminus S^2 \simeq D^3 \times S^2$$

The second one

$$(D^3 \times S^2) \setminus (D^3 \times \{y_0\}) \simeq D^3 \times D^2 \cong D^5$$

$\square$

Thus we have another homotopy equivalent representation of the space X^5.

$$X^5 \simeq X''^5 = D^5 \cup_{f''} S^2 \vee S^3 \tag{5.35}$$

For some map $f'' \in \pi_4(S^2 \vee S^3)$. We have already shown that $\pi_4(S^2 \vee S^3) \cong \pi_4(S^2) \oplus \pi_4(S^3) \oplus \pi_5(S^2 \times S^3, S^2 \vee S^3)$. The attaching map in Y^5 is an element $(1, 0, 1)$ from this groups. We need to figure out that our attaching map f'' is the representative of the same element to prove the Theorem 5.4.1.

It will be helpful to understand how the group $\pi_4(S^2 \vee S^3)$ behaves under maps induced by collapsing individual spheres $p_2 : S^2 \vee S^3 \to S^2$ and $p_3 : S^2 \vee S^3 \to S^3$. In particular, we would like to see what happens to the individual summands $\pi_4(S^2) \oplus \pi_4(S^3) \oplus \pi_5(S^2 \times S^3, S^2 \vee S^3)$ under these projections.

Lemma 5.4.3. *For $i, j \in \{2, 3\}$ the maps $\pi_4(S^i) \xrightarrow{\iota_{i*}} \pi_4(S^2 \vee S^3) \xrightarrow{p_{j*}} \pi_4(S^j)$ are isomorphisms if $i = j$ and zero if $i \neq j$.*

Proof. For induced maps we have $p_{j_*}\iota_{i_*} = (p_j\iota_i)_* : \pi_4(S^i) \to \pi_4(S^j)$. The map $p_j\iota_i : S^i \to S^j$ is constant if $i \neq j$ and an identity if $i = j$. The Lemma follows. $\square$

Lemma 5.4.4. *For $i \in \{2, 3\}$ the composition $\pi_5(S^2 \times S^3, S^2 \vee S^3) \xrightarrow{\partial} \pi_4(S^2 \vee S^3) \xrightarrow{p_{i*}} \pi_4(S^i)$ is zero.*

Proof. The map p_{i*} is composition $\pi_4(S^2 \vee S^3) \xrightarrow{i_*} \pi_4(S^2) \oplus \pi_4(S^3) \xrightarrow{\text{proj.}} \pi_4(S^i)$ where i_* is the map from the short exact sequence 5.30. From the exactness of the short exact sequence the result follows. $\square$

Lemma 5.4.5. *For the map $f'' = (m, n, k) \in \pi_4(S^2) \oplus \pi_4(S^3) \oplus \pi_5(S^2 \times S^3, S^2 \vee S^3) \cong \pi_4(S^2 \vee S^3)$ we have $n = 0 \in \pi_4(S^3)$.*

Proof. Let $p_3 : S^2 \vee S^3 \to S^3$ be the collapse map. This map induces a homomorphism on the homotopy groups $p_{3*} : \pi_4(S^2 \vee S^3) \to \pi_4(S^3)$. We already understand this induced homomorphism, namely

$$p_{3*} : \pi_4(S^2) \oplus \pi_4(S^3) \oplus \pi_5(S^2 \times S^3, S^2 \vee S^3) \to \pi_4(S^3)$$

$$\begin{aligned} (1,0,0) &\mapsto 0 && \text{by the Lemma 5.4.3} \\ (0,1,0) &\mapsto 1 && \text{by the Lemma 5.4.3} \\ (0,0,1) &\mapsto 0 && \text{by the Lemma 5.4.4} \end{aligned}$$

To finish the proof, one need only to show that $p_{3*}(f'') = p_3 \circ f'' = 0 \in \pi_4(S^3)$.

Factoring out S^2 from the space X''^5 gives the space

$$D^5 \cup_{p_{3*}(f'')} S^3 \tag{5.36}$$

Similarly let us take the space X'^5 where we factor out S^2 from the right side of its characterisation 5.34.

We obtain

$$X'^5/S^2 = \left(D^3 \times S^2 \cup_{f'} S^2\right)/S^2 = D^3 \times S^2 \cup_{f_0} * \tag{5.37}$$

for a constant map $f_0 : \partial D^3 \times S^2 \to *$.

This brings us to the familiar space $S^2 \times S^3$. We already know that it is a 5-cell attached to $S^2 \vee S^3$ by the map $[\iota_2, \iota_3] \in \pi_4(S^2 \vee S^3)$, the Whitehead product. And the space in 5.37 is thus exactly $S^2 \times S^3/S^2$. Recall that the characteristic map of the cell attached by the Whitehead map was the generator of $\pi_5(S^2 \times S^3, S^2 \vee S^3)$. But this group is sent to zero under the map p_{3*}. Hence also $p_{3*}(f'') = 0$.

□

We will need the following

Lemma 5.4.6. *The space $S^k \times S^l$ for $k, l \geq 1$ has the following cohomology*

$$\textit{If } k \neq l \textit{ we have } H^i(S^k \times S^l) = \begin{cases} \mathbb{Z} & \textit{if } i{=}0,\ k,\ l,\ k{+}l \\ 0 & \textit{otherwise} \end{cases} \tag{5.38}$$

$$\textit{If } k = l \textit{ we have } H^i(S^k \times S^l) = \begin{cases} \mathbb{Z} & \textit{if } i{=}0,\ 2k \\ \mathbb{Z} \oplus \mathbb{Z} & \textit{if } i{=}k \\ 0 & \textit{otherwise} \end{cases} \tag{5.39}$$

and the cup product of the generators of the k and l cohomology gives the generator of the top cohomology $e^k \smile e^l = \pm e^{k+l}$, sign depends on the choice of the generators.

Proof. An application of the Künneth Theorem [5, 3.15, p. 216] gives an isomorphism of rings.

$$\begin{aligned} H^*(S^k) \otimes H^*(S^l) &\xrightarrow{\cong} H^*(S^k \times S^l) \\ (a \otimes b) &\mapsto (a \times b) \end{aligned} \tag{5.40}$$

Generators of the ring on the right are $1 \otimes 1$, $e^k \otimes 1$, $1 \otimes e^l$, $e^k \otimes e^l$ and multiplication is

$$(e^k \otimes 1) \cdot (1 \otimes e^l) = e^k \otimes e^l \mapsto \pm e^{k+l} \in H^{k+l}(S^k \times S^l)$$

$$(e^k \otimes 1) \cdot (e^k \otimes 1) = (e^k \cdot e^k) \otimes 1 = 0$$

$$(1 \otimes e^l) \cdot (1 \otimes e^l) = 1 \otimes (e^l \cdot e^l) = 0$$

□

The following Theorem is from a 1957 paper by I. M. James [8, Theorem 4.1]

Theorem 5.4.7. *Let X be a CW complex, $p, q \geq 2$ be integers such that $H^{p+q-1}(X)$ is finite. Suppose we have elements $a \in H^p(X)$ and $b \in H^q(X)$ such that $a \smile b = 0$. Take an element $\lambda \in \pi_{p+q-1}(X)$. Let us now attach a $(p+q)$-cell with λ. $X^* = X \cup_\lambda D^{p+q}$ and let $c \in H^{p+q}(X^*)$ be the element corresponding to this new cell.*

Then there are unique $a' \in H^p(X^*)$ and $b' \in H^q(X^*)$ which map to a and b respectively under map induced by inclusion $X \to X^*$. Then there exists an integer m such that $a' \smile b' = mc$.

Define a function $h : \pi_{p+q-1}(X) \to \mathbb{Z}$ by sending λ to m. This function is a homomorphism.

Calculation 5.4.8. Assuming Theorem 5.4.7 we apply it to our problem. In the Theorem let X be $S^2 \vee S^3$, $p = 2$, $q = 3$. The fourth cohomology of $S^2 \vee S^3$ is obviously vacuous, hence finite as required. Let $a = e^2 \in H^2(S^2 \vee S^3)$ and $b = e^3 \in H^3(S^2 \vee S^3)$ be the elements of the appropriate cellular cohomologies representing the duals of the 2-cell and the 3-cell respectively. The cup product $e^2 \smile e^3$ is zero because the fifth cohomology of $S^2 \vee S^3$ is trivial. The Theorem implies that $h : \pi_4(S^2) \oplus \pi_4(S^3) \oplus \pi_5(S^2 \times S^3, S^2 \vee S^3) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z} \to \mathbb{Z}$ is a homomorphism.

Recall that product of spheres $S^k \times S^l$ was a $(k+l)$-cell attached to $S^k \vee S^l$. From the Lemma 5.4.6 we know the cup product $e^k \smile e^l = \pm 1 \cdot e^{k+l}$. From the Lemma 5.3.3 we know that the generator of the group $\pi_5(S^2 \times S^3, S^2 \vee S^3)$ is the characteristic map of this cell e^{k+l}. Hence $h(0,0,1) = \pm 1$. This (up to a sign) completely defines h by sending $(m,n,k) \overset{h}{\mapsto} \pm k$. For the space X''^5 to be a Poincaré complex and so to have Poincaré duality we need $e^2 \smile e^3$ to be a generator. This is because if $e^2 \smile e^3 = me^5$ for an integer m, then by the basic properties of the cup/cap products [12, p. 336] we would have

$$m = (me^5) \frown e_5 = (e^2 \smile e^3) \frown e_5 = e^2 \frown (e^3 \frown e_5) = e^2 \frown (\pm e^2) = \pm 1 \quad (5.41)$$

where we use that $e^3 \frown e_5 = \pm e^2$ from the Poincaré duality and e_2, e_3 and e_5 are the elements in the respective homologies corresponding to the cohomology elements e^2, e^3 and e^5 respectively. This makes sense because the cellular chain and and cochain complex have all boundary and coboundary maps zero.

$$\cdots \to \mathbb{Z} \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \to \mathbb{Z} \quad (5.42)$$

Hence we need $k = \pm 1$.

What we have learned can be explicitly stated: the presence or absence of torsion elements : $\pi_4(S^2)$, $\pi_4(S^3)$ in the attaching maps from $\pi_4(S^2 \vee S^3)$ does not plays any role in the cup product of the space X''^5.

Proof of the Theorem 5.4.1. We know from previous calculations that f'' was of the form $(m, 0, \pm 1) \in \pi_4(S^2) \oplus \pi_4(S^3) \oplus \pi_5(S^2 \times S^3, S^2 \vee S^3)$. That leaves us with two options for f'' either $(1, 0, \pm 1)$ or $(0, 0, \pm 1)$. But the latter option would make X''^5 homotopy equivalent to a manifold $S^2 \times S^3$ which we know it is not, because X''^5 does not have a vector bundle reduction of its Spivak normal fibration.

Therefore $f'' \simeq \eta^2 \pm [\iota_2, \iota_3]$. If we have a minus sign, precompose f'' with a map of degree -1 and then we have $f'' \simeq \eta^2 + [\iota_2, \iota_3]$ since $(-\eta^2 = \eta^2)$. □

The reader may wonder what kind of space would one obtain for different choices of the attaching map in $D^5 \cup_g S^2 \vee S^2$. To consider only Poincaré complexes we take $[\iota_2, \iota_3]$ with unit multiplicity in the summand of g.

Note 5.4.9. Let us take the space $Y = D^5 \cup_g S^2 \vee S^3$ for some $g = (m, n, \pm 1) \in \pi_4(S^3) \oplus \pi_4(S^2) \oplus \pi_5(S^2 \times S^3, S^2, \vee S^3)$. Then

i) For $g = (0, 0, \pm 1)$ we have that $Y \simeq S^2 \times S^3$.

ii) For $g = (1, 0, \pm 1)$ we of course have $Y \simeq X''^5$.

iii) For $g = (0, 1, \pm 1)$ or $g = (1, 1, \pm 1)$ we actually get homotopy equivalent results since, as indicated here [10, p. 32], there is a homotopy equivalence

$$(\iota_2, \iota_2 \circ \eta + \iota_3) : S^2 \vee^3 \to S^2 \vee^3 \tag{5.43}$$

which sends $(0, 1, \pm 1)$ to $(1, 1, \pm 1)$.

Proof of the Theorem 5.4.7. [8, Lemma 4.2] Consider three elements $\lambda_1, \lambda_2, \lambda_3 \in \pi_{p+q-1}(X)$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 0$. If we proved that $h(\lambda_1) + h(\lambda_2) + h(\lambda_3) = 0$ the Theorem would follow. This is because

$$\begin{aligned} h(\lambda_1) + h(\lambda_2) + h(\lambda_3) &= 0 \\ h(\lambda_1) + h(\lambda_2) + h(-\lambda_1 - \lambda_2) &= 0 \\ h(\lambda_1) + h(\lambda_2) &= h(\lambda_1 + \lambda_2) \end{aligned}$$

where the relation $h(-\lambda) = -h(\lambda)$ used in the last step holds because both correspond to changing the sign of the chosen generator of the cell attached by λ.

We will use an indexing integer $t = 1, 2, 3$. Denote X_t which is obtained by attaching a $(p+q)$-cell to X by λ_t. Denote c_t the cohomology class in $H^{p+q}(X_t)$ corresponding to this cell.

Let $X' = X_1 \cup_X X_2 \cup_X X_3$ be the union of the three spaces, where the original X are glued together. Let us have the long exact sequence of cohomology for a pair (X', X)

$$\cdots \longrightarrow H^r(X', X) \longrightarrow H^r(X') \xrightarrow{i_t^* j_t^*} H^r(X) \xrightarrow{\partial} H^{r+1}(X', X) \longrightarrow \cdots \tag{5.44}$$

Where we denote the inclusions $X \overset{i_t}{\hookrightarrow} X_t \overset{j_t}{\hookrightarrow} X'$.

If we consider $r = p$, then $a \in H^p(X)$ and $\partial a = 0$ since $X' \setminus X$ only contain $(p+q)$-cells and from $q \geq 2$ we have $H^{p+1}(X', X) = 0$ and also $H^p(X', X) = 0$. Therefore a has a unique preimage a' in $H^p(X')$. Taking $r = q$ we can find a unique preimage b' of b in $H^q(X')$ using the same argument and an assumption that $p \geq 2$.

Now the cup product $a' \smile b'$ is in the cohomology $H^{p+q}(X')$. The elements c_1, c_2, c_3 can be thought of as generators of the cohomology $H^{p+q}(X', X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. There are integers m_1, m_2, m_3 such that $a' \smile b' = m_1 c_1 + m_2 c_2 + m_3 c_3$.

We now use the naturality of the cup product with respect to the previously defined maps $j_t : X_t \to X'$.

$$j_t^*(a') \smile j_t^*(b') = j_t^*(a' \smile b') = j_t^*(m_1c_1 + m_2c_2 + m_3c_3) = m_t j_t^*$$

But by definition of the map h we have $h(\lambda_t) = m_t$.

We will need a map $f : S^{p+q} \to X'$ which is of degree one on each of the attached cells (i.e. for each $t = 1, 2, 3$ the composition $S^{p+q} \xrightarrow{f} X' \to X'/(\cup_{i=1,2,3;i\neq t} X_i) \cong S^{p+q}$ is of degree one (for an orientation preserving homeomorphism $X'/(\cup_{i=1,2,3;i\neq t} X_i) \cong S^{p+q}$)). The existence of such map is proven in the Lemma 5.4.10 bellow.

Assume we have such a map f. For simplicity of notation take $t = 1$. Since $X'/X_2 \cup X_3$ does not have any $(p+q-1)$-cells we have $H_{p+q-1}(X'/X_2 \cup X_3) = 0$. From the naturality of the Universal coefficient Theorem for cohomology [5, p. 201] we have

$$\begin{tikzcd} H^{p+q}(S^{p+q}) \arrow[r, "\cong"] & \mathrm{Hom}(H_{p+q}(S^{p+q}), \mathbb{Z}) \\ H^{p+q}(X'/X_2 \cup X_3) \arrow[u, "f^*"] \arrow[r, "\cong"] & \mathrm{Hom}(H_{p+q}(X'/X_2 \cup X_3), \mathbb{Z}) \arrow[u, "(f_*)^*"'] \end{tikzcd} \tag{5.45}$$

The group $H_{p+q}(X'/X_2 \cup X_3)$ is isomorphic to $\mathbb{Z}$ and by the degree one property $(f_*)^*$ induces an isomorphism. Hence $f^* : H^{p+q}(X'/X_2 \cup X_3) \to H^{p+q}(S^{p+q})$ is an isomorphism.

Element $c_1 \in H^{p+q}(X'/X_2 \cup X_3)$ is the generator and hence is mapped to a generator $f^*(c_1) = c$ of $H^{p+q}(S^{p+q})$. The elements c_2, c_3 are mapped to the same generator $f^*(c_2) = c$, $f^*(c_3) = c$. We know that $f^*(a') = 0$ and $f^*(b') = 0$ because the cohomology of the sphere is nontrivial only in dimensions 0 and $p + q$.

We have by naturality

$$\begin{aligned} 0 = f^*(j_t^*(a')) \smile f^*(j_t^*(b')) &= f^*(j_t^*(a') \smile j_t^*(b')) \\ &= f^*(m_1c_1 + m_2c_2 + m_3c_3) = (m_1 + m_2 + m_3)c \end{aligned} \tag{5.46}$$

Since $c \neq 0$ we must have $m_1 + m_2 + m_3 = 0$, and so $h(\lambda_1) + h(\lambda_2) + h(\lambda_3) = 0$. ☐

Lemma 5.4.10. *In the situation of the proof of the Theorem 5.4.7 there is the map $f : S^{p+q} \to X'$ as required in the text.*

Proof. Let us write S^{p+q} as the union of the upper and the lower hemisphere $S^{p+q} = D_+^{p+q} \cup_{S^{p+q-1}} D_-^{p+q}$. We shall define f for each component separately. Take the upper hemisphere with its boundary (D^{p+q}_+, S^{p+q-1}). Embed in it two disks $(p+q-1)$-disks with boundary by pointed maps i_1, i_2 (see Figure 5.4)

$$\begin{aligned} i_1 &: (D^{p+q-1}, S^{p+q-2}) \hookrightarrow (D^{p+q}_+, S^{p+q-1}) \\ i_2 &: (D^{p+q-1}, S^{p+q-2}) \hookrightarrow (D^{p+q}_+, S^{p+q-1}) \end{aligned} \tag{5.47}$$

such that $\mathrm{Im}(i_1) \cap \mathrm{Im}(i_2) = \{x_0\}$ the basepoint $x_0 \in S^{p+q-1} \subset D^{p+q}_+$. See Figure 5.4.

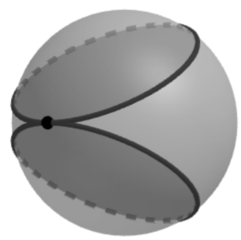

Figure 5.1: Two embedded disks

Define the map $P : (D^{p+q}_+, S^{p+q-1}) \to (D^{p+q} \vee D^{p+q} \vee D^{p+q}, S^{p+q-1} \vee S^{p+q-1} \vee S^{p+q-1})$ by factoring out the union of the embedded disks $\mathrm{Im}(i_1) \cup \mathrm{Im}(i_2)$.

Let ϕ_t be the characteristic map of the cell attached by the fixed representative of the class λ_t, for $t = 1, 2, 3$. Now define the map $f_+ : (D^{p+q}_+, S^{p+q-1}) \to X'$ by the composition $(\phi_1 \vee \phi_2 \vee \phi_3, \lambda_1 \vee \lambda_2 \vee \lambda_3) \circ P$.

Take the lower hemisphere (D^{p+q}_-, S^{p+q-1}). Recall that the sum $\lambda_1 + \lambda_2 + \lambda_3$ is trivial in $\pi_{p+q-1}(X)$, hence is nullhomotopic in X. Choose such a null-homotopy $H : S^{p+q-1} \times I \to X$ that $H_0 = \lambda_1 + \lambda_2 + \lambda_3$ (for the same representatives of the classes λ_t as were used above) and $H_1 = *$. Hence H factors through $S^{p+q-1} \times I / S^{p+q-1} \times \{1\} \cong CS^{p+q-1} \cong D^{p+q}$.

$$\begin{tikzcd} S^{p+q-1} \times I \arrow[rr, "H"] \arrow[dr] & & X \arrow[r] & X' \\ & D^{p+q} \arrow[ur, "H'"] & & \end{tikzcd} \tag{5.48}$$

Define $f_- : (D^{p+q}_-, S^{p+q-1}) \to X'$ to be H' in such a way that f_- restricts to H_0 on the subspace S^{p+q-1}. This way f_+ and f_- agree on S^{p+q-1} and hence we can combine them to define $f : S^{p+q} \to X'$.

To show the degree requirement, taking $t = 1$ for simplicity, observe that the composition $f' : S^{p+q} \to X' \to X'/X_2 \cup X_3$ is homotopic to the map $\phi_1' : S^{p+q} \to X'/X_2 \cup X_3$ induced by the characteristic map $\phi_1 : D^{p+q} \to X'$ of the cell attached by the map λ_1. The argument works analogously for $t = 2, 3$. $\square$

Bibliography

[1] Diarmuid Crowley, Wolfgang Lück, and Tibor Macko. Surgery Theory : Foundations. http://www.mat.savba.sk/~macko/surgery-book.html, 2020. [Not yet published, version: December-2020].

[2] James F. Davis and Paul Kirk. Lecture Notes in Algebraic Topology. *Algebraic Topology*, page 169, 2001.

[3] Glen E. Bredon. *Topology and Geometry.* Graduate Texts in Mathematics, Springer-Verlag, New York, 1. edition, 1993.

[4] Daniel Henry Gottlieb. Poincaré Duality and Fibrations. *Proceedings of the American Mathematical Society*, Vol. 76(No. 1):148–150, 1979.

[5] Allan Hatcher. *Algebraic topology.* Cambridge University Press; 1st edition, 2001.

[6] Allen Hatcher. Vector Bundles and K-Theory. https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf, 2017. [Available online; accessed April-2021].

[7] Morris W. Hirsch. *Differential Topology.* Graduate Texts in Mathematics, Springer-Verlag, New York, 1976.

[8] I. M. James. Note on Cup-Products. *Proceedings of the American Mathematical Society*, 8(2):374–383, 1957.

[9] Wolfgang Lück. *Algebraische Topologie: Homologie und Mannigfaltigkeiten.* Vieweg Studium -Aufbaukurs Mathematik, Vieweg, 2005.

[10] Ib Madsen and R James Milgram. *The classifying spaces for surgery and cobordism of manifolds*, volume (AM-92). Princeton University Press, 1979.

[11] John W. Milnor and James D. Stacheff. *Characteristic Classes. (AM-76), Volume 76.* Annals of Mathematics Studies: Princeton University Press, 1974.

[12] Andrew Ranicki. *Algebraic and Geometric Surgery.* Oxford Mathematical Monograph, Oxford, 2007.

[13] Colin Patrick Rourke and Brian Joseph Sanderson. *Introduction to piecewise-linear topology.* Springer-Verlag, New York, 1972.

[14] Michael Spivak. Spaces Satisfying Poincaré Duality. *Topology*, 6(April 1965):77–101, 1967.

[15] James Stasheff. A classification theorem for fibre spaces. *Topology*, 2(3):239–246, 1963.

[16] Mark Steinberger and James West. Covering homotopy propertiesof maps between C.W. complexes or ANR's. *Proceedings of the American Mathematical Society*, 92(4):573–577, 1984.

[17] Jeffrey Strom. *Modern Classical Homotopy Theory.* American Mathematical Society, Province Rhode, Island, 2011.

[18] George W. Whitehead. *Elements of Homotopy Theory.* Graduate Texts in Mathematics, Springer-Verlag, New York, 1. edition, 1978.